LAST NAME: SOLUTIONS

FIRST NAME: ____________

STUDENT NUMBER: ____________

MIDTERM

McGill University

Faculty of Science

Math 133 - Vectors, Matrices and Geometry

Instructor: E. Richer

Date: May 22nd 2008

INSTRUCTIONS:

You have 1.5 hours to complete this midterm.

The midterm comprises 6 pages including this cover page.

The midterm has five questions.

the midterm is marked out of a TOTAL of 50 MARKS.

No books are permitted.

SHOW AND JUSTIFY ALL YOUR WORK.

Question	Mark
1	
2	
3	
4	
5	
Total	

Question 1. *(10 marks)*
Determine all values of a and b for which the following system:
$2x - y + z = 3$
$x + 2y - az = 22$
$4x + 3y - z = b$
has
(i) a unique solution
(ii) infinitely many solutions
(iii) no solutions

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 1 & 2 & -a & 22 \\ 4 & 3 & -1 & b \end{array}\right] \xrightarrow{2R_2 \to R_2} \left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 2 & 4 & -2a & 44 \\ 4 & 3 & -1 & b \end{array}\right]$$

$R_2 - R_1 \to R_2$
$R_3 - 2R_1 \to R_3$
$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 5 & -2a-1 & 41 \\ 0 & 5 & -3 & b-6 \end{array}\right]$$

$R_3 - R_2 \to R_3$
$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 0 & 5 & -2a-1 & 41 \\ 0 & 0 & 2a-2 & b-47 \end{array}\right]$$

(i) ONE SOLUTION
RANK OF COEFFICIENT MATRIX $= 3$
$2a - 2 \neq 0$
$\boxed{a \neq 1}$

(ii) INFINITELY MANY SOLUTIONS (ROW OF ZEROS)
$2a - 2 = 0$
& $b - 47 = 0$
$\boxed{\begin{array}{l} a = 1 \\ b = 47 \end{array}}$

(iii) NO SOLUTIONS
$2a - 2 = 0$
& $b - 47 \neq 0$
$\boxed{\begin{array}{l} a = 1 \\ b \neq 47 \end{array}}$

Question 2. *(10 marks)*
(a) Consider a set U in $\mathbb{R}^n$. State the three conditions that must be satisfied in order for U to be a subspace of $\mathbb{R}^n$.

(b) Is $U_1 = \left\{ \begin{bmatrix} s \\ t \\ 1 \end{bmatrix} \middle| s,t \text{ in } \mathbb{R} \right\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

(c) Is $U_2 = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| x,y,z \text{ in } \mathbb{R} \text{ and } 2x-y+z=0 \right\}$ a subspace of $\mathbb{R}^3$? Justify your answer.

(a) (1) $\vec{0}$ is in U
(2) IF $\vec{x}$ & $\vec{y}$ are in U then $\vec{x}+\vec{y}$ is in U
(3) IF $\vec{x}$ is in U, K A scalar, then $K\vec{x}$ is in U

(b) U_1 is Not A subspace of $\mathbb{R}^3$
ALL three conditions FAIL

(c) ① $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is in U_2 because $2(0)-0+0=0$

② Let $V_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ & $V_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be in U_2

then $2x_1 - y_1 + z_1 = 0$ & $2x_2 - y_2 + z_2 = 0$

Now $V_1 + V_2 = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}$ Check if V_1+V_2 is in U_2

$$2(x_1+x_2) - (y_1+y_2) + z_1+z_2$$
$$= (2x_1-y_1+z_1) + (2x_2-y_2+z_2)$$
$$= 0 + 0 = 0$$

so V_1+V_2 is in U_2.

③ Let K be A scalar
then $KV_1 = \begin{bmatrix} Kx_1 \\ Ky_1 \\ Kz_1 \end{bmatrix}$ Check if KV_1 is in U_2

$$2(Kx_1) - Ky_1 + Kz_1$$
$$= K(2x_1 - y_1 + z_1) = K(0) = 0$$

so KV_1 is in U_2. U_2 is A subspace.

Question 3. *(10 marks)*

Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 4 \\ -1 & 4 & 3 & 7 \\ 0 & 2 & 2 & 4 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Given that R is the reduced echelon form of the matrix A, answer the following:

(a) Find the dimension of the row space of A and determine a basis for it.

(b) Find the dimension of the column space of A and determine a basis for it.

(c) Find the dimension of the null space of A and determine a basis for it.

(a) dim Row A = # OF NON-ZERO rows = 2

BASIS = $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$

(b) dim col A = # OF LEADING 1'S = 2

BASIS = $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \\ 2 \end{bmatrix} \right\}$

(c) dim (null A) = # parameters = 2

Let $x_3 = s$, $x_4 = t$

$x_2 = -x_3 - 2x_4$
$= -s - 2t$

$x_1 = -x_3 - x_4$
$= -s - t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-t \\ -s-2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} t$$

BASIS = $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

Question 4. *(10 marks)*
Suppose $\{v_1, v_2, v_3\}$ are linearly independent vectors.
(a) Is the set $\{v_1, v_2, v_1 - v_2\}$ linearly independent? Justify your answer.
(b) If $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, $w_3 = v_3 + v_1$, show that $\{w_1, w_2, w_3\}$ is linearly independent.

(a) No. Clearly the third vector is a linear combination (non-trivial) of the first two.

(b) Let $aw_1 + bw_2 + cw_3 = 0$

then $a(v_1+v_2) + b(v_2+v_3) + c(v_3+v_1) = 0$

$\Rightarrow (a+c)v_1 + (a+b)v_2 + (b+c)v_3 = 0$

Since v_1, v_2, v_3 are linearly ind, then coefficients must be zero

$$\left.\begin{array}{l} a+c=0 \\ a+b=0 \end{array}\right\} \Rightarrow \begin{array}{l} c-b=0 \\ c=b \end{array}$$

$b+c=0 \Rightarrow$

$$c+c=0$$
$$2c=0$$
$$c=0$$

$\Rightarrow b=0$

$\Rightarrow a=0$

Only the trivial solution $a=b=c=0$ satisfies $aw_1 + bw_2 + cw_3 = 0$

So $\{w_1, w_2, w_3\}$ is linearly ind.

Question 5. *(10 marks)*

(a) Compute $\det A$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 3 \\ -2 & -2 & 3 \end{bmatrix}$

Is A invertible? Justify your answer.

(b) B and C are 3 x 3 matrices with $\det B$ = -2 and $\det C = 4$. Compute $\det(4B^3C^tB^{-1})$.

(a) WE EXPAND ALONG row 1

$$\det A = 1 \det\begin{bmatrix} 1 & 3 \\ -2 & 3 \end{bmatrix} + 1 \det\begin{bmatrix} 3 & 1 \\ -2 & -2 \end{bmatrix}$$

$$= 1\,(3+6) + 1\,(-6+2)$$

$$= 9 - 4 = 5$$

since $\det A \neq 0$ then A is invertible

(b) $\det(4B^3C^tB^{-1})$

$$= 4^3 (\det B)^3 (\det C) \frac{1}{\det B}$$

$$= 4^3 (-2)^3 (4) \frac{1}{-2}$$

$$= 4^5$$