Math 133 Midterm Solutions.

Problem 1: *The problem is asking to find <u>all</u> the values of a and b so that a given system system of equations has either no, a unique, or infinitely many solutions.*

Perform Gaussian row-reduction:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 6 \\ 1 & 2 & -a & 11 \\ 4 & 3 & -1 & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -a & 11 \\ 2 & -1 & 1 & 6 \\ 4 & 3 & -1 & b \end{array}\right] \to$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -a & 11 \\ 0 & -5 & 2a+1 & -16 \\ 0 & -5 & 4a-1 & b-44 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -a & 11 \\ 0 & -5 & 2a+1 & -16 \\ 0 & 0 & 2a-2 & b-28 \end{array}\right].$$

In the first step we interchange the first two rows; in the second step we subtract twice the first row from the second row, and subtract four times the first row from the third row; in the third step we subtract the second row from the third row. In principle we can also divide the second row by -5 to get an explicit 1 in the second column.

If $2a - 2 \neq 0$ (in other words $a \neq 1$) then we can divide the third row by $2a - 2$ and obtain a 1 in the third column. Hence the system has a unique solution regardless of what values are on the right. (Indeed, we can uniquely solve for x_3 using the third equation, then we can uniquely solve for x_2 using the second equations, and finally we can uniquely solve for x_1 using the first equations.)

If $2a-2 = 0$ AND $b \neq 28$ then the system is inconsistent: the third equation is of the form $0x_1 + 0x_2 + 0x_3 = b - 28 \neq 0$. In this case the system has no solutions.

If $2a-2 = 0$ AND $b = 28$ then the third row is a row of zeroes. The system is consistent as there are no rows with zero coefficients and nonzero value on the right, and indeterminate as the variable x_3 is free (there is no pivot in the third column). Thus the system has infinitely many solutions (paramaterized by x_3).

Problem 2: *The problem is asking to write A^{-1} and A as the product of elementary matrices.*

First we row-reduce the matrix A:

$$A = \begin{bmatrix} 0 & 9 \\ 5 & -4 \end{bmatrix} \to \begin{bmatrix} 5 & -4 \\ 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -4/5 \\ 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -4/5 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

In the first step we switch the two rows; in the second step we divide the first row by 5; in the third step we divide the second row by 9; and in the fourth step we add 4/5 of the second row to the first row.

Each such row operation corresponds to left-multiplication by a suitable elementary matrix. Therefore $E_4E_3E_2E_1A = I$, where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/9 \end{bmatrix}, \; E_4 = \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix}.$$

a).

$$A^{-1} = E_4E_3E_2E_1 = \begin{bmatrix} 1 & 4/5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/9 \end{bmatrix} \begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is the product of elementary matrices.

b). $A = (A^{-1})^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$.

Computing the inverses of the elementary matrices (which are themselves elementary), we find that

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & -4/5 \\ 0 & 1 \end{bmatrix}$$

is the product of elementary matrices.

Remark: there are several different ways to do row-reduction, and so different answers are possible.

Problem 3:

a). If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, then $AB = \begin{bmatrix} a_{11}x & a_{12}y \\ a_{21}x & a_{22}y \end{bmatrix}$.

b). If $B = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, then $B^2 = \begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix}$, and so the condition $B^2 = B$ means that $\begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$. It follows that $x^2 = x$ and $y^2 = y$. Now, if $x^2 = x$, then $x^2 - x = 0$, so $x(x-1) = 0$, and so there are exactly two solutions for x: $x = 0$ and $x = 1$. Similarly there are exactly two solutions for y: $y = 0$ and $y = 1$. Combining these $2 \times 2 = 4$ cases, we get the following possibities for B: $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (as the hint suggests, there are exactly 4 of such solutions).

Remark: For part b) it is not enough to just guess four different solutions for B, you need to show why these are <u>all</u> such solutions.

Problem 4:

a). The definition that C *is the inverse of* A is: $AC = I$, $CA = I$.

b). <u>First solution:</u> according to the definition, to prove that $B^{-1}A^{-1}$ is the inverse of AB, one needs to check that $(B^{-1}A^{-1})(AB) = I$, and that $(AB)(B^{-1}A^{-1}) = I$.

We have: $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. Similarly, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$.

Second solution: if C denotes the inverse of AB, one can find C as follows: since $C(AB) = I$, then $(CA)B = I$. Multiplying both sides by B^{-1} from the right, we get $(CA)BB^{-1} = IB^{-1}$, so $(CA)I = B^{-1}$, and so $CA = B^{-1}$. Multiplying both sides by A^{-1} from the right, we get $CAA^{-1} = B^{-1}A^{-1}$ and so $C = B^{-1}A^{-1}$. Thus we have found that the inverse C of AB is $B^{-1}A^{-1}$.

Problem 5:

$$\begin{vmatrix} g & h & i \\ 7a-3g & 7b-3h & 7c-3i \\ 2d & 2e & 2f \end{vmatrix} = \begin{vmatrix} g & h & i \\ 7a & 7b & 7c \\ 2d & 2e & 2f \end{vmatrix} = 7\cdot 2 \begin{vmatrix} g & h & i \\ a & b & c \\ d & e & f \end{vmatrix} =$$

$$= -7\cdot 2 \begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix} = 7\cdot 2 \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7\cdot 2\cdot 5 = 70.$$

In the first step we add three times the first row to the second row (having no effect on the determinant); in the second step we pull the multiple of 7 out of the second row and a multiple of 2 out of the third row; in the third step we interchange the first two rows (multiplying the determinant by -1); and in the last step we interchange the last two rows (multiplying the determinant by another -1).

Remark: for a matrix A the symbol "$|A|$" means precisely "the determinant of A".

Problem 6:

a).

$$\det(2B^TA^{-1}B^{-1}C^2AC^{-1}) = 2^3\det(B^TA^{-1}B^{-1}C^2AC^{-1}) =$$
$$= 2^3\det(B^T)\det(A^{-1})\det(B^{-1})\det(C)^2\det(A)\det(C^{-1}) =$$
$$= 2^3\det B\frac{1}{\det A}\frac{1}{\det B}\det(C)^2\det(A)\frac{1}{\det C} = 2^3\det C = 56.$$

b). We have: $\operatorname{adj}(A) = \det(A)\cdot A^{-1} = -5A^{-1}$. Hence

$$\det(\operatorname{adj}(A)) = \det(-5A^{-1}) = (-5)^3\det(A^{-1}) = \frac{(-5)^3}{\det A} = (-5)^2 = 25.$$

Remark: for part b) one can also use the formula $\det(\operatorname{adj}(A)) = (\det A)^{n-1}$, where n denote the size of the matrix A.