

Test 3

This test is graded out of 50 marks. No books, notes, no graphing calculator or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (5 marks) Determine the indeterminate form, then evaluate the limit, using L'Hôpital Rule if necessary:

$$\lim_{x \rightarrow \infty} \frac{\ln x^3}{x^2} \quad (\text{the indeterminate form is } \frac{\infty}{\infty})$$

We can therefore use L'Hôpital's Rule

$$= \lim_{x \rightarrow \infty} \frac{\frac{1}{x^3} \cdot 3x^2}{2x}$$

$$= \lim_{x \rightarrow \infty} \frac{3}{2x^2}$$

$$= 0$$

Question 2. (5 marks) Determine the indeterminate form, then evaluate the limit, using L'Hôpital Rule if necessary:

$$\lim_{x \rightarrow \infty} x^2 e^{-3x} \quad (\text{the indeterminate form is } \frac{\infty}{\infty})$$

We need to manipulate the expression in order to use L'Hôpital's Rule.

$$= \lim_{x \rightarrow \infty} \frac{x^2}{e^{3x}} \quad \text{the indeterminate form is } \frac{\infty}{\infty} \therefore \text{we use L'Hôpital's Rule.}$$

$$= \lim_{x \rightarrow \infty} \frac{2x}{3e^{3x}} \quad \text{the indeterminate form is } \frac{\infty}{\infty} \therefore \text{we use L'Hôpital's Rule again.}$$

$$= \lim_{x \rightarrow \infty} \frac{2}{9e^{3x}}$$

$$= 0$$

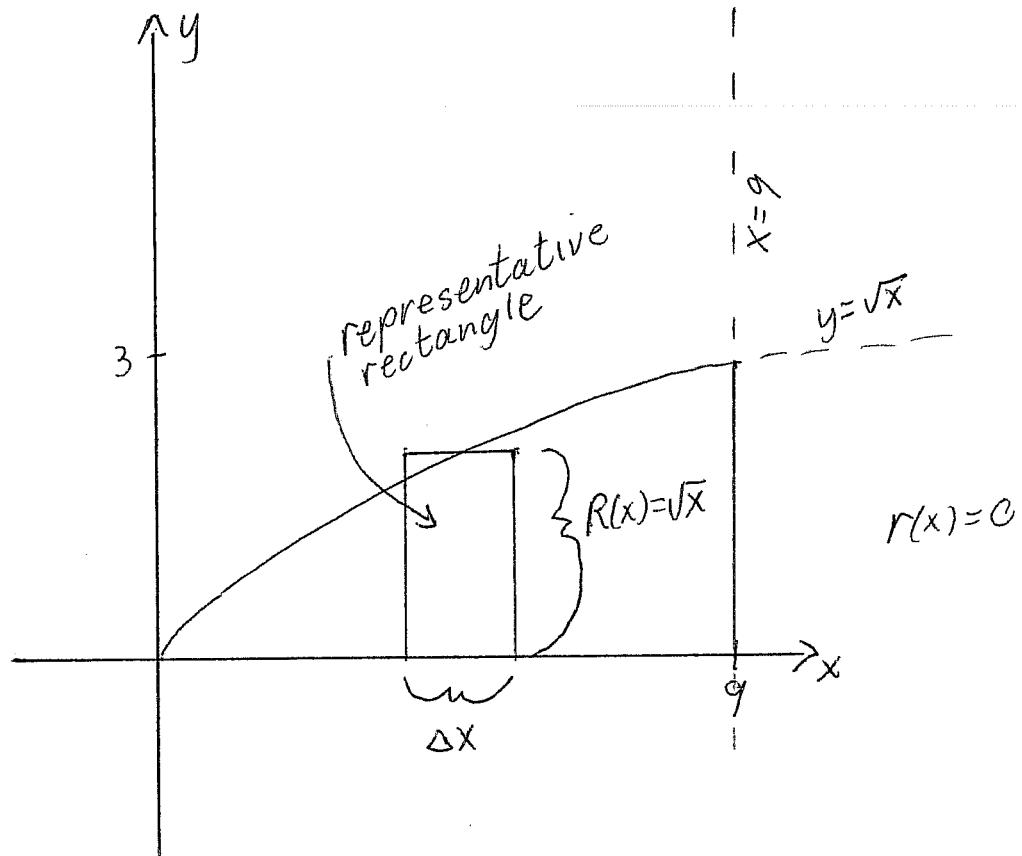
Question 3. (5 marks) Integrate the following improper integral if it converges:

$$\begin{aligned}
 & \int_0^\infty xe^{-x/3} dx \\
 &= \lim_{b \rightarrow \infty} \int_0^b xe^{-x/3} dx \quad \text{Integration by parts} \\
 &\quad u = x \quad du = dx \\
 &\quad v = -3e^{-x/3} \quad dv = e^{-x/3} dx \\
 &= \lim_{b \rightarrow \infty} \left[-3xe^{-x/3} \Big|_0^b + 3 \int_0^b e^{-x/3} dx \right] \\
 &= \lim_{b \rightarrow \infty} \left[-3be^{-b/3} + 3(0)e^{0/3} + 3 \left[-3e^{-x/3} \right]_0^b \right] \\
 &= \lim_{b \rightarrow \infty} \left[\frac{-3b}{e^{b/3}} + 3(3)e^{-b/3} - 9e^{0/3} \right] \\
 &= 9 + \lim_{b \rightarrow \infty} \frac{-3b}{e^{b/3}} \quad \text{has indeterminate form } \frac{-\infty}{\infty} \\
 &= 9 + \lim_{b \rightarrow \infty} \frac{-3(3)}{e^{b/3}} \quad \text{we then use L'Hospital's Rule} \\
 &= 9
 \end{aligned}$$

Question 4. (5 marks) Integrate the following improper integral if it converges:

$$\begin{aligned} & \int_0^2 \frac{1}{\sqrt[4]{2-x}} dx \\ &= \lim_{b \rightarrow 2^-} \int_0^b \frac{1}{\sqrt[4]{2-x}} dx \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{4(\sqrt[4]{2-b})^3}{3} + \frac{4(\sqrt[4]{2-0})^3}{3} \right] \\ &= \lim_{b \rightarrow 2^-} \left[-\frac{4(\sqrt[4]{2-b})^3}{3} + \frac{4(\sqrt[4]{2})^3}{3} \right] \\ &= \frac{4}{3} (\sqrt[4]{2})^3 \end{aligned}$$

Question 5. (5 marks) Find the volume of the solid generated by revolving the region bounded by the graphs of the equations: $y = \sqrt{x}$, $y = 0$, $x = 9$ about the x -axis.



Representative element: $\Delta V = \pi [(R(x))^2 - (r(x))^2] \Delta x$

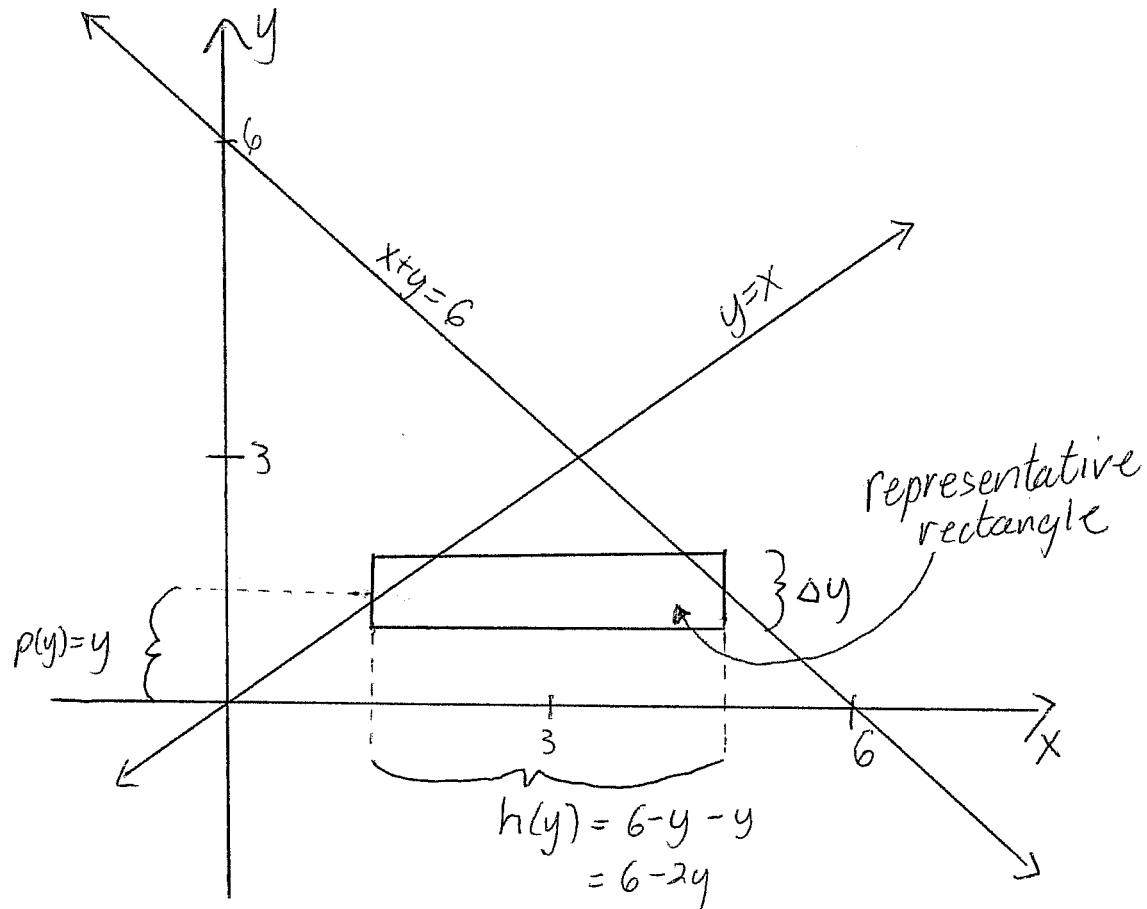
$$= \pi [x] \Delta x$$

$$\int_0^9 \pi x dx = \frac{\pi x^2}{2} \Big|_0^9$$

$$= \frac{81\pi}{2}$$

∴ the volume is $\frac{81\pi}{2}$

Question 6. (5 marks) Find the volume of the solid generated by revolving the region bounded by the graphs of the equations: $x+y=6$, $y=x$, $y=0$ about the x -axis.



Representative element: $\Delta V = 2\pi p(y)h(y) \Delta y$

$$\Delta V = 2\pi y(6-2y) \Delta y$$

$$\begin{aligned}
 \int_0^3 2\pi y(6-2y) dy &= 2\pi \int_0^3 6y - 2y^2 dy \\
 &= 2\pi \left[3y^2 - \frac{2y^3}{3} \right]_0^3 \\
 &= 2\pi \left[3(3)^2 - 2 \frac{(3)^3}{3} \right] \\
 &= 2\pi [27 - 18] \\
 &= 2\pi [9] \\
 &= 18\pi
 \end{aligned}$$

∴ the volume is 18π

Question 7. (5 marks) Find the arc length of the graph of the function $y = \frac{x^4}{10} + \frac{1}{6x^2}$ over the interval $[1, 2]$.

$$S = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \quad \frac{dy}{dx} = \frac{4x^3}{10} - \frac{2}{6x^3}$$

$$= \int_1^2 \sqrt{1 + \left(\frac{4x^3}{10} - \frac{2}{6x^3}\right)^2} dx$$

$$= \int_1^2 \sqrt{1 + \left(\frac{2x^3}{5} - \frac{1}{3x^3}\right)^2} dx$$

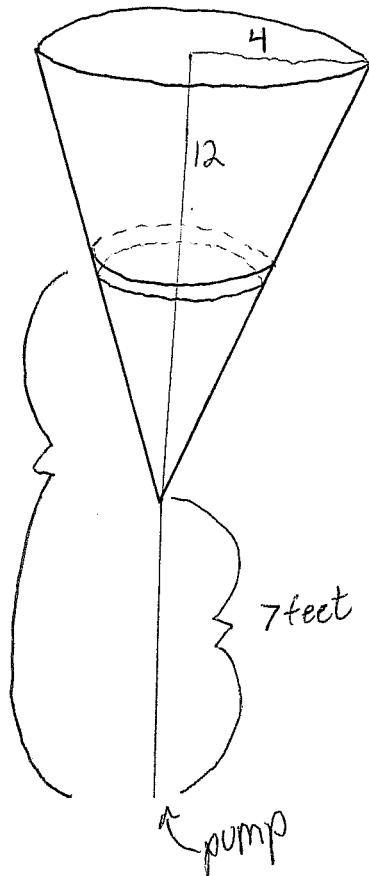
$$= \int_1^2 \sqrt{1 + \frac{4x^6}{25} - \frac{4}{15} + \frac{1}{9x^6}} dx$$

$$= \int_1^2 \sqrt{\frac{4x^6}{25} + \frac{11}{15} + \frac{1}{9x^6}} dx$$

$$= \int_1^2 \sqrt{\left(\frac{2x^3}{5} + \right)}$$

Question 8. (5 marks) An open tank has the shape of a circular cone with its tip oriented downward. The tank is 8 feet across the top and 12 feet high. How much work is done in filling the tank by pumping the water from 7 feet below the tank? (The water weighs 62.4 pounds per cubic foot)

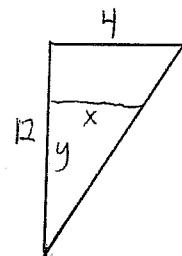
distance of slice $y+7$



Volume of Slice:

$$\Delta V = \pi x^2 \Delta y$$

but we want the volume of the slice with respect to y . Therefore we set up the following ratio.



$$\frac{x}{y} = \frac{4}{12}$$

$$x = \frac{1}{3}y$$

$$\therefore \Delta V = \pi \left(\frac{1}{3}y\right)^2 \Delta y$$

$$\Delta V = \pi \frac{y^2}{9} \Delta y$$

$\Delta F = \Delta V$ (weight of water per cubic foot)

$$\Delta F = 62.4 \left(\pi \frac{y^2}{9}\right) \Delta y$$

$$\Delta F = \frac{104\pi}{15} y^2 \Delta y$$

Work to move slice:

$$\Delta W = \Delta F \times \text{distance}$$

$$\Delta W = \frac{104\pi}{15} y^2 (y+7) \Delta y$$

$$\text{Work} = \int_0^{12} \frac{104\pi}{15} (y^2 + 7y^3) dy$$

$$= \left[\frac{104\pi}{15} \left(\frac{y^4}{4} + 7 \frac{y^4}{3} \right) \right]_0^{12}$$

$$= \frac{104\pi}{15} \left(\frac{12^4}{4} + 7 \frac{(12)^3}{3} \right)$$

$$= \underline{\underline{319488\pi}} \quad \therefore 200740 \text{ lb-ft}$$

Question 9. (5 marks) Integrate the following indefinite integral:

$$\int \frac{3x^2-2}{x^3+x} dx = \int \frac{3x^2-2}{x(x^2+1)} dx$$

Solve using partial fractions.

$$\frac{3x^2-2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$3x^2-2 = A(x^2+1) + (Bx+C)x$$

$$\text{Let } x=0$$

$$-2 = A$$

$$\text{Let } x=1$$

$$3(1)^2-2 = A(1^2+1) + (B(1)+C)(1)$$

$$1 = -2(2) + B+C$$

$$5 = B+C$$

$$5-B = C \quad (1)$$

$$\text{Let } x=-1$$

$$3(-1)^2-2 = A(-1)^2+1 + (B(-1)+C)(-1)$$

$$1 = -2(2) + B-C$$

$$5 = B-C \quad (2)$$

Using equation ① and ②

$$5 = B - (5-B)$$

$$5 = 2B-5$$

$$5 = B$$

$$\therefore C = 0$$

$$\int \frac{-2}{x} + \frac{5x}{x^2+1} dx = -2\ln|x| + \frac{5}{2} \ln|x^2+1| + C$$

Question 10. (5 marks) Integrate the following indefinite integral:

$\int x\sqrt{1+x^2} dx$ by trig. substitution or substitution.

Let $u = 1+x^2$

$du = 2x dx$

$\frac{du}{2} = x dx$

$$\textcircled{1}, \textcircled{2} \quad \int \sqrt{u} \frac{du}{2} = \frac{u^{3/2}}{3} + C$$

$$= \frac{(1+x^2)^{3/2}}{3} + C$$

Bonus Question. (3 marks) Evaluate the following limit:

$$\lim_{x \rightarrow 1^+} (\ln x)^{x-1} \quad \text{Let } y = \lim_{x \rightarrow 1^+} (\ln x)^{x-1} \quad \text{then apply ln on both sides}$$

$$\ln y = \ln \lim_{x \rightarrow 1^+} (\ln x)^{x-1}$$

$$\ln y = \lim_{x \rightarrow 1^+} \ln (\ln x)^{x-1}$$

$$\ln y = \lim_{x \rightarrow 1^+} (x-1) \ln (\ln x) \quad \text{has indeterminate form } 0 \cdot -\infty$$

$$\ln y = \lim_{x \rightarrow 1^+} \frac{\ln (\ln x)}{\frac{1}{(x-1)}} \quad \text{has indeterminate form } \frac{0}{0}$$

$$\ln y = \lim_{x \rightarrow 1^+} \frac{\frac{1}{\ln x} \frac{1}{x}}{-\frac{1}{(x-1)^2}}$$

$$\ln y = \lim_{x \rightarrow 1^+} \frac{-\frac{1}{(x-1)^2}}{\frac{1}{(\ln x)x}} \quad \text{has indeterminate form } \frac{0}{0}$$

$$\ln y = \lim_{x \rightarrow 1^+} \frac{-2(x-1)}{\ln x + \frac{x}{x}}$$

$$\ln y = \lim_{x \rightarrow 1^+} \frac{-2(x-1)}{\ln x + 1}$$

$$\ln y = 0$$

$$y = 1$$

$$\therefore \lim_{x \rightarrow 1^+} (\ln x)^{x-1} = 1$$