Dawson College: **Calculus 1**: 201-NYA-05-C3: Winter 2008: Saturday April 19th

Name: SOLUTIONS

Student ID: ______________

TEST 3

This test is marked out of 70 marks. No books, notes, graphing calculators, or cell phones allowed. SHOW ALL YOUR WORK.

Question 1. *(4 marks each)*
Find the derivative.
(a) $f(x) = \ln(\ln(x^2+1))$
(b) $f(x) = e^x(\sin x + \cos x)$
(c) $f(x) = \cos(5^{2x})$
(d) $f(x) = e^x \log_3(x+2)$

(a) $f'(x) = \dfrac{1}{\ln(x^2+1)} \cdot \dfrac{1}{x^2+1} \cdot 2x$

$= \dfrac{2x}{(x^2+1)\ln(x^2+1)}$

(b) $f'(x) = e^x(\sin x + \cos x) + (\cos x - \sin x)e^x$

$= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$

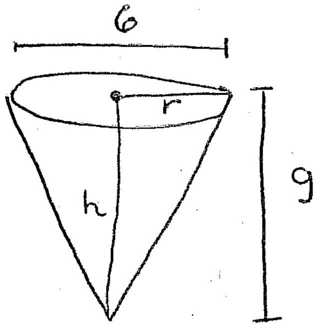
$= 2e^x \cos x$

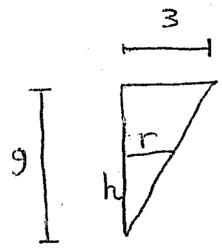
(c) $f'(x) = -\sin(5^{2x}) \cdot 5^{2x} \ln 5 \cdot 2$

(d) $f'(x) = e^x \log_3(x+2) + \dfrac{1}{\ln 3 (x+2)} e^x$

$= e^x\left(\log_3(x+2) + \dfrac{1}{\ln 3 (x+2)}\right)$

Question 2. *(10 marks)*
Water is being pumped into a conical tank. The tank has height 9m and the diameter of the top is 6m. If the water level is rising at a rate of 0.3m/min when the height of the water is 2m, find the rate at which water is being pumped into the tank.
(Formula: Volume of a cone = $(1/3)\pi r^2 h$ where r is the radius of the base of the cone and h is the height of the cone)





$$\frac{r}{h} = \frac{3}{9}$$

$$\frac{r}{h} = \frac{1}{3}$$

$$\boxed{r = h/3}$$

Looking for $\frac{dV}{dt}$ when $h=2$

Given $dh/dt = 0.3$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{3}\right)^2 h$$

$$V = \frac{\pi}{27} h^3 \implies \frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{9}(2)^2(0.3)$$

$$= 0.133\pi \ m^3/min$$

Question 3. *(3 marks)*
Find y' by implicit differentiation, given $\ln y = e^{x+y}$.

$$\frac{1}{y}y' = e^{x+y}(1+y')$$

$$y' = y(e^{x+y}) + yy'e^{x+y}$$

$$y' - yy'e^{x+y} = ye^{x+y}$$

$$y'(1-ye^{x+y}) = ye^{x+y}$$

$$\boxed{y' = ye^{x+y} / 1-ye^{x+y}}$$

Question 4. *(7 marks)*
Find the derivative of $y = (\sin x)^{1/x}$.

Use log differentiation

$$\ln y = \ln((\sin x)^{1/x})$$

$$\ln y = \frac{1}{x}\ln \sin x = \frac{\ln \sin x}{x}$$

$$\frac{1}{y}y' = \frac{\left(\frac{1}{\sin x}\cos x\right)x - \ln \sin x}{x^2}$$

$$y' = y\left(\frac{(\cot x)x - \ln \sin x}{x^2}\right)$$

$$\boxed{y' = (\sin x)^{1/x}\left[\frac{(\cot x)x - \ln \sin x}{x^2}\right]}$$

Question 5. *(5 marks)*
Evaluate the following infinite limits.
(a) $\lim_{x\to-\infty} \frac{x^3+2x-1}{x+x^2-12x^3}$
(b) $\lim_{x\to\infty} \frac{x^2}{x^5+4}$

(a) $\lim_{x\to-\infty} \frac{x^3+2x-1}{x+x^2-12x^3} = \boxed{\frac{1}{-12}}$

(b) $\lim_{x\to\infty} \frac{x^2}{x^5+4} = \boxed{0}$

Question 6 *(5 marks)*
Find the absolute minimum and maximum of the function $g(x) = \frac{1}{3}x^3 + 4x^2 - 9x + 12$ on the interval $[-5,2]$

$g'(x) = x^2 + 8x - 9$
$= (x+9)(x-1)$

CRITICAL #s
$x=-9$ & $x=1$

$x=-9$ is Not in the interval

so

$g(1) = \frac{1}{3}(1) + 4(1) - 9(1) + 12$
$= 7\frac{1}{3}$

$g(2) = \frac{1}{3}(2)^3 + 4(2)^2 - 9(2) + 12$
$= 8/3 + 16 - 18 + 12 = 8/3 + 10$
$= 12\frac{2}{3}$

$g(-5) = \frac{1}{3}(-5)^3 + 4(-5)^2 - 9(-5) + 12$
$= -125/3 + 157 = 115\frac{1}{3}$

Abs MAX is $115\frac{1}{3}$ at $x=-5$

Abs MIN is $7\frac{1}{3}$ at $x=1$

Question 7. *(5 marks)*
Use the second derivative test to find and classify the relative extrema of
$f(x) = 2x^3 - 21x^2 - 48x + 6$.

$$f'(x) = 6x^2 - 42x - 48$$
$$= 6(x^2 - 7x - 8)$$
$$= 6(x-8)(x+1)$$

Critical #s $x = 8$ & $x = -1$

$$f''(x) = 12x - 42$$

$$f''(8) = 12(8) - 42$$
$$= 54 \quad +$$

Min.

so f has a rel min at $x = 8$

$$f''(-1) = -12 - 42$$
$$= -54 \quad -$$

Max.

so f has a rel max at $x = -1$

Question 8 *(15 marks)*
Sketch the graph of $f(x) = \frac{x}{x^2-1}$.
The first and second derivatives are: $f'(x) = -\frac{x^2+1}{(x^2-1)^2}$ and $f''(x) = \frac{2x(x^2+2)}{(x^2-1)^3}$
Clearly identify on the sketch:
(a) The x and y intercepts
(b) Any relative extrema
(c) The intervals where $f(x)$ is concave up and where it is concave down
(d) Any points of inflection
(e) Any horizontal and vertical asymptotes

(a) WHEN $x=0$ $y=0$
WHEN $y=0$ $x=0$

(0,0) is the x & y intercept

(b) CRITICAL points:
$f'(x)$ doesn't exist when $x=\pm 1$

INTerval	$-\infty < x < -1$	$-1 < x < 1$	$1 < x < \infty$
test	-2	0	2
sign of f'	$-$	$-$	$-$
incr./dec	↘	↘	↘

THE FUNCTION is ALWAYS decreasing

(c) $f''(x)=0$ when $x=0$
$f''(x)$ doesn't exist when $x=\pm 1$

INTERVAL	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
test	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Sign of f''	$-$	$+$	$-$	$+$
concavity	$\cap$	$\cup$	$\cap$	$\cup$

(d) THERE is AN inflection point At $(0,0)$
The concavity changes At the two V.As $x=\pm 1$ but they are not actual points on the graph of f

(e) V.As when den. of $f = 0$
NUM $\neq 0$
THis happens At $x=\pm 1$

H.As $\lim_{x\to\pm\infty} \frac{x}{x^2-1} = 0$

so $y=0$ is AN H.A.