Dawson College: **Calculus 1**: 201-NYA-05-C3: Winter 2008: Sat. March 15th

Name:
Student ID: SOLUTIONS

TEST 2

This test is graded out of 45 marks (there are 3 bonus marks available). No books, notes, graphing calculators or CELL PHONES are allowed! SHOW ALL YOUR WORK. If you need more space for your answer use the back of the page.

Question 1. *(4 marks)*
Find the derivative of $f(x) = 1 - x^2$ using the LIMIT PROCESS.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(1-(x+\Delta x)^2) - (1-x^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1 - x^2 - 2x\Delta x - \Delta x^2 - 1 + x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(-2x - \Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} -2x - \Delta x = \boxed{-2x}$$

Question 2. *(5 marks)*
Find an equation of the tangent line to the graph of f at the point $(-1, 1)$.

$$f(x) = x\left(1 - \frac{4}{x+3}\right)$$

$$f(x) = x - \frac{4x}{x+3} = \frac{x(x+3) - 4x}{x+3} = \frac{x^2 + 3x - 4x}{x+3}$$

$$= \frac{x^2 - x}{x+3}$$

Now we use quotient rule to find

$$f'(x) = \frac{(2x-1)(x+3) - (1)(x^2 - x)}{(x+3)^2} = \frac{2x^2 + 5x - 3 - x^2 + x}{(x+3)^2}$$

$$= \frac{x^2 + 6x - 3}{(x+3)^2}$$

THE slope of the line is given by $f'(-1)$

$$f'(-1) = \frac{(-1)^2 + 6(-1) + (-3)}{(-1+3)^2} = \frac{1-9}{4} = \frac{-8}{4} = -2$$

So we have slope -2 & equation

$$y = -2x + b$$

Use the point $(-1,1)$ to find b

$$1 = -2(-1) + b \qquad b = 1-2$$
$$b = -1$$

THE equation is thus

$$\boxed{y = -2x - 1}$$

Question 3. *(2 marks each)*
Find the derivative of the function and simplify your answer where possible.

(a) $y = \frac{2x^{5/2}}{3x}$

(b) $y = x\sin x + \cos x$

(c) $y = \frac{7}{3x^3}$

(d) $y = \frac{\sec t}{t}$

(e) $y = \frac{x+1}{\sin x}$

(a) $y = \frac{2}{3}\frac{x^{5/2}}{x} = \frac{2}{3}x^{3/2}$

so $y' = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right)$

$\boxed{= x^{1/2}}$

(b) $y' = (\sin x + (\cos x)x) - \sin x$

$= \boxed{x\cos x}$

(c) $y = \frac{7}{3x^3} = \frac{7}{3}x^{-3}$

so $y' = \frac{7}{3}(-3x^{-4}) = \boxed{-\frac{7}{x^4}}$

(d) $y = \frac{\sec t}{t} = \frac{1}{t\cos t} = (t\cos t)^{-1}$

$y' = -1(t\cos t)^{-2}(\cos t - (-\sin t)t)$

$= \boxed{\frac{-1}{(t\cos t)^2}(\cos t + t\sin t)}$

(e) $y' = \boxed{\frac{1(\sin x) - \cos x(x+1)}{\sin^2 x}}$

Question 4.

Find the derivative of the function and simplify your answer where possible.

(a) *(4 marks)*
$f(x) = (\frac{3x^2-2}{2x+3})^5$

(b) *(3 marks)*
$g(t) = \sin(t^2 \cos t)$

(c) *((4 marks))*
$f(t) = \frac{1}{2}t^2\sqrt{16-t^2}$

(d) *(5 marks)*
$h(x) = 3\tan^4(x^2+5x)$

(a) CHAIN RULE & quotient rule

$$f'(x) = 5\left(\frac{3x^2-2}{2x+3}\right)^4\left(\frac{6x(2x+3) - 2(3x^2-2)}{(2x+3)^2}\right)$$

$$= 5\left(\frac{3x^2-2}{2x+3}\right)^4\left(\frac{12x^2+18x-6x^2+4}{(2x+3)^2}\right)$$

$$\boxed{= \frac{5(3x^2-2)^4}{(2x+3)^6}(6x^2+18x+4)}$$

(b) $g'(t) = \cos(t^2\cos t)\left(2t\cos t - (\sin t)t^2\right)$

(CHAIN & PRODUCT RULE)

(c) PRODUCT & CHAIN RULE

$$f'(t) = \frac{1}{2}(2t)\sqrt{16-t^2} + \frac{1}{2}(16-t^2)^{-1/2}(-2t)\left(\frac{1}{2}t^2\right)$$

$$= \boxed{t\sqrt{16-t^2} - \frac{1}{2}t^3\left(\frac{1}{\sqrt{16-t^2}}\right)}$$

(d) $h(x) = 3(\tan(x^2+5x))^4$

(CHAIN RULE + CHAIN RULE)

(SHORT WAY): USING $\frac{d}{dx}(\tan x) = \sec^2(x)$

$$\boxed{h'(x) = 12\tan^3(x^2+5x) \cdot \sec^2(x^2+5x) \cdot (2x+5)}$$

Long way: Using $\tan x = \dfrac{\sin x}{\cos x}$

$$h(x) = 3\tan^4(x^2+5x)$$

$$= 3\left(\frac{\sin(x^2+5x)}{\cos(x^2+5x)}\right)^4$$

$$h'(x) = 12\left(\frac{\sin(x^2+5x)}{\cos(x^2+5x)}\right)^3 \cdot \left(\frac{\cos(x^2+5x)(2x+5)\cdot\cos(x^2+5x) - {} - \sin(x^2+5x)(2x+5)\cdot\sin(x^2+5x)}{\cos^2(x^2+5x)}\right)$$

$$= 12\tan^3(x^2+5x)\cdot\left(\frac{(2x+5)\left[(\cos^2(x^2+5x) + \sin^2(x^2+5x)\right]}{\cos^2(x^2+5x)}\right)$$

$$\boxed{= \frac{12\tan^3(x^2+5x)}{\cos^2(x^2+5x)}\Big((2x+5)(1)\Big)}$$

Question 5 *(4 marks)*
Find dy/dx by implicit differentiation then evaluate the derivative at the point $(-1,1)$.
$(x+y)^3 = x^3 + y^2$

$$3(x+y)^2(1+y') = 3x^2 + 2yy'$$

$$3(x+y)^2 + 3(x+y)^2 y' = 3x^2 + 2yy'$$

$$3(x+y)^2 y' - 2yy' = 3x^2 - 3(x+y)^2$$

$$y'\left(3(x+y)^2 - 2y\right) = 3x^2 - 3(x+y)^2$$

$$y' = \frac{3x^2 - 3(x+y)^2}{\left(3(x+y)^2 - 2y\right)}$$

At $(-1,1)$

$$y' = \frac{3(-1)^2 - 3(0)^2}{3(0)^2 - 2(1)}$$

$$= \boxed{3/-2}$$

Question 6. *(6 marks)*
Find d^2y/dx^2 in terms of x and y.
$1 - xy = y^2$

First find y': $0 - (y + y'x) = 2yy'$

$$-y = 2yy' + xy'$$

$$-y = (2y+x)y' \Rightarrow \boxed{y' = \frac{-y}{2y+x}}$$

Now derive y' (quotient rule) to find y''

$$y'' = \frac{-y'(2y+x) - (2y'+1)(-y)}{(2y+x)^2} = \frac{-2yy' - xy' + 2yy' + y}{(2y+x)^2}$$

$$= \frac{y'(-2y - x + 2y) + y}{(2y+x)^2}$$

substitute $y' = \dfrac{-y}{2y+x}$

$$y'' = \frac{\left(\frac{-y}{2y+x}\right)(\quad -x) \quad + y}{(2y+x)^2}$$

$$= \frac{\frac{+xy \ + \ y(2y+x)}{2y+x}}{(2y+x)^2}$$

$$= \frac{+xy + 2y^2 + xy}{(2y+x)^3}$$

$$= \frac{+2y^2 + 2xy}{(2y+x)^3} = \boxed{\frac{2(+y^2 + xy)}{(2y+x)^3}}$$

Question 7 *(4 marks)*
Find an equation of the line that is tangent to the graph of f and parallel to the line $y = 7x - 1$. Where $f(x) = x^2 + 3x - 1$.

$f'(x) = 2x + 3$ the slope of the tangent must be the same as the slope of $y = 7x - 1$ because the lines are parallel

$2x + 3 = 7$ $\quad$ $2x = 4$
$x = 2$

$x = 2$ so $y = f(2) = 2^2 + 3(2) - 1$
$= 4 + 6 - 1$
$= 9$

so at the point $(2, 9)$ the tangent line has slope 7.

Equation: $y = 7x + b$
$9 = 7(2) + b$ $\quad$ $b = -5$

so $\boxed{y = 7x - 5}$

BONUS QUESTION *(3 marks)*
Find the $f^{(400)}(x)$ (the 400th derivative of f) where $f(x) = \sin x + 2$

Find the pattern

$f'(x) = \cos x$ $\quad$ $f^5(x) = \cos x$
$f^2(x) = -\sin x$ $\quad$ $f^6(x) = -\sin x$
$f^3(x) = -\cos x$ $\quad$ $f^7(x) = -\cos x$
$\boxed{f^4(x) = \sin x}$ $\quad$ $\boxed{f^8(x) = \sin x}$

Every 4th derivative we return to $\sin x$.
Since 400 is a multiple of 4

$\boxed{f^{400}(x) = \sin x}$