L'Hôpital's Rule

In what follows a can represent a [finite] real number or ∞ or $-\infty$.

Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. Then $\lim_{x\to a}\frac{f(x)}{g(x)}$ may or may not exist, and we call this limit an **indeterminate form of type** $\frac{0}{0}$.

Similarly, suppose $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Then $\lim_{x\to a}\frac{f(x)}{g(x)}$ may or may not exist, and we call this limit an **indeterminate form of type** $\frac{\infty}{\infty}$.

L'Hopital's Rule Suppose f and g are differentiable functions and that $\lim_{x\to a}\frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$ **or** $\frac{\infty}{\infty}$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists (or is ∞ or $-\infty$).

NOTE – There are other types of limits for which L'Hôpital's Rule can be applied. But in these notes we will only consider the two types mentioned above.

Example 1 Find $\lim_{x\to 2}\frac{5x^3-13x^2+6x}{4x^2-13x+10}$

Since $\lim_{x\to 2}(5x^3-13x^2+6x) = 0$ and $\lim_{x\to 2}(4x^2-13x+10) = 0$ we can apply L'Hôpital's Rule.

$$\lim_{x\to 2}\frac{5x^3-13x^2+6x}{4x^2-13x+10} = \lim_{x\to 2}\frac{15x^2-26x+6}{8x-13} = \frac{15(2)^2-26(2)+6}{8(2)-13} = \frac{14}{3}$$

Example 2 Find $\lim_{x\to\infty}\frac{10x+5}{3x^2-7x+4}$

Since $\lim_{x\to\infty}(10x+5) = \infty$ and $\lim_{x\to\infty}(3x^2-7x+4) = \infty$ we can apply L'Hôpital's Rule.

$$\lim_{x\to\infty}\frac{10x+5}{3x^2-7x+4} = \lim_{x\to\infty}\frac{10}{6x-7} = 0$$

Example 3 Find $\lim_{x\to 0} \frac{e^x}{1-\cos x}$

Since $\lim_{x\to 0}\left(e^x\right)=1$ and $\lim_{x\to 0}\left(1-\cos x\right)=0$ we can **not** apply L'Hôpital's Rule.

$$\lim_{x\to 0} \frac{e^x}{1-\cos x} = \infty$$

Example 4 Find $\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1}$

Since $\lim_{x\to 0}\left(\cos x - 1\right)=0$ and $\lim_{x\to 0}\left(e^x - 1\right)=0$ we can apply L'Hôpital's Rule.

$$\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1} = \lim_{x\to 0} \frac{-\sin x}{e^x} = \frac{-\sin(0)}{e^{(0)}} = 0$$

Example 5 Find $\lim_{x\to 1^+} \frac{7\sqrt{x-1}}{\sin(x-1)}$

Since $\lim_{x\to 1^+} 7\sqrt{x-1} = 0$ and $\lim_{x\to 1^+} \sin(x-1) = 0$ we can apply L'Hôpital's Rule.

$$\lim_{x\to 1^+} \frac{7\sqrt{x-1}}{\sin(x-1)} = \lim_{x\to 1^+} \frac{\left[\frac{7}{2\sqrt{x-1}}\right]}{\left[\cos(x-1)\right]} = \lim_{x\to 1^+} \frac{7}{2\cdot\cos(x-1)\cdot\sqrt{x-1}} = \infty$$

Example 6 Find $\lim_{x\to\infty} \frac{3\ln(5x+3)}{2\ln(x+4)}$

Since $\lim_{x\to\infty} 3\ln(5x+3) = \infty$ and $\lim_{x\to\infty} 2\ln(x+4) = \infty$ we can apply L'Hôpital's Rule.

$$\lim_{x\to\infty} \frac{3\ln(5x+3)}{2\ln(x+4)} = \lim_{x\to\infty} \frac{\left[\frac{15}{5x+3}\right]}{\left[\frac{2}{x+4}\right]} = \lim_{x\to\infty} \frac{15x+60}{10x+6}$$

The limit on the right is also indeterminate (type $\frac{\infty}{\infty}$). We can apply L'Hôpital's Rule again.

$$\lim_{x\to\infty} \frac{3\ln(5x+3)}{2\ln(x+4)} = \lim_{x\to\infty} \frac{\left[\frac{15}{5x+3}\right]}{\left[\frac{2}{x+4}\right]} = \lim_{x\to\infty} \frac{15x+60}{10x+6} = \lim_{x\to\infty} \frac{15}{10} = \frac{15}{10} = \frac{3}{2}$$

EXERCISES Find each limit. Use L'Hôpital's Rule where appropriate. Otherwise use a more elementary method.

1. $\lim\limits_{x\to 3}\dfrac{2x^2-3x-9}{x^2-2x-3}$

2. $\lim\limits_{x\to \infty}\dfrac{4x^3+x-3}{x^2-5x+8}$

3. $\lim\limits_{x\to 1}\dfrac{2x^3-x^2-4x+3}{3x^3-5x^2+x+1}$

4. $\lim\limits_{x\to \infty}\dfrac{6x-5}{4x^2+7x+9}$

5. $\lim\limits_{x\to 0}\dfrac{3x^2+8x}{5x^3}$

6. $\lim\limits_{x\to \infty}\dfrac{x^2-7x-10}{6x^2-x-1}$

7. $\lim\limits_{x\to 0}\dfrac{1-e^x}{2x}$

8. $\lim\limits_{x\to 0}\dfrac{x^2+x}{\sin 3x}$

9. $\lim\limits_{x\to 0^+}\dfrac{\sin x}{1-\cos x}$

10. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sin x}$

11. $\lim\limits_{x\to 0}\dfrac{\sin 2x}{\sin x}$

12. $\lim\limits_{x\to 0}\dfrac{e^x-x-1}{5x^2}$

13. $\lim\limits_{x\to \infty}\dfrac{e^{3x}}{\ln x}$

14. $\lim\limits_{x\to 1}\dfrac{x-1}{\sqrt{x}-1}$

15. $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{1-\cos x}$

16. $\lim\limits_{x\to \infty}\dfrac{\sqrt{x-1}}{4x+5}$

17. $\lim\limits_{x\to 1}\dfrac{\ln x}{\sin(x-1)}$

18. $\lim\limits_{x\to \infty}\dfrac{\ln(x+1)}{\sqrt{x}}$

19. $\lim\limits_{x\to \infty}\dfrac{e^x+x}{\ln x}$

20. $\lim\limits_{x\to 1}\dfrac{e^{x-1}-1}{(x-1)^3}$

21. $\lim\limits_{x\to \infty}\dfrac{\ln(x-10)}{\ln(4x+1)}$

22. $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\ln(x+1)}$

23. $\lim\limits_{x\to \infty}\dfrac{e^{4x}}{e^{3x}+x}$

24. $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{e^{5x}-1}$

SOLUTIONS

1. $\lim_{x\to 3}\frac{2x^2-3x-9}{x^2-2x-3}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 3}\frac{4x-3}{2x-2} = \frac{9}{4}$

2. $\lim_{x\to\infty}\frac{4x^3+x-3}{x^2-5x+8}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{12x^2+1}{2x-5}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{24x}{2} = \infty$

3. $\lim_{x\to 1}\frac{2x^3-x^2-4x+3}{3x^3-5x^2+x+1}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 1}\frac{6x^2-2x-4}{9x^2-10x+1}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 1}\frac{12x-2}{18x-10} = \frac{10}{8} = \frac{5}{4}$

4. $\lim_{x\to\infty}\frac{6x-5}{4x^2+7x+9}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{6}{8x+7} = 0$

5. $\lim_{x\to 0}\frac{3x^2+8x}{5x^3}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{6x+8}{15x^2} = \infty$

6. $\lim_{x\to\infty}\frac{x^2-7x-10}{6x^2-x-1}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{2x-7}{12x-1}$ $\left(\text{type }\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty}\frac{2}{12} = \frac{2}{12} = \frac{1}{6}$

7. $\lim_{x\to 0}\frac{1-e^x}{2x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{-e^x}{2} = -\frac{1}{2}$

8. $\lim_{x\to 0}\frac{x^2+x}{\sin 3x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{2x+1}{3\cos 3x} = \frac{1}{3}$

9. $\lim_{x\to 0^+}\frac{\sin x}{1-\cos x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0^+}\frac{\cos x}{\sin x} = \infty$

10. $\lim_{x\to 0}\frac{1-\cos x}{\sin x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{\sin x}{\cos x} = 0$

11. $\lim_{x\to 0}\frac{\sin 2x}{\sin x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{2\cos 2x}{\cos x} = 2$

12. $\lim_{x\to 0}\frac{e^x-x-1}{5x^2}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{e^x-1}{10x}$ $\left(\text{type }\frac{0}{0}\right)$

$= \lim_{x\to 0}\frac{e^x}{10} = \frac{1}{10}$

13. $\lim_{x\to\infty} \frac{e^{3x}}{\ln x}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{\left[3e^{3x}\right]}{\left[\frac{1}{x}\right]}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} 3xe^{3x} = \infty$

14. $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 1} \frac{[1]}{\left[\frac{1}{2\sqrt{x}}\right]} = 2$

15. $\lim_{x\to 0^+} \frac{\sqrt{x}}{1-\cos x}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 0^+} \frac{\left[\frac{1}{2\sqrt{x}}\right]}{[\sin x]}$

$= \lim_{x\to 0^+} \frac{1}{2\sqrt{x}\sin x} = \infty$

16. $\lim_{x\to\infty} \frac{\sqrt{x-1}}{4x+5}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{\left[\frac{1}{2\sqrt{x-1}}\right]}{[4]}$

$= \lim_{x\to\infty} \frac{1}{8\sqrt{x-1}} = 0$

17. $\lim_{x\to 1} \frac{\ln x}{\sin(x-1)}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 1} \frac{\left[\frac{1}{x}\right]}{[\cos(x-1)]} = 1$

18. $\lim_{x\to\infty} \frac{\ln(x+1)}{\sqrt{x}}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{\left[\frac{1}{x+1}\right]}{\left[\frac{1}{2\sqrt{x}}\right]}$

$= \lim_{x\to\infty} \frac{2\sqrt{x}}{x+1}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{\left[\frac{1}{\sqrt{x}}\right]}{[1]} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

19. $\lim_{x\to\infty} \frac{e^x + x}{\ln x}$ $\left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{\left[e^x+1\right]}{\left[\frac{1}{x}\right]}$

$= \lim_{x\to\infty} x\left(e^x+1\right) = \infty$

20. $\lim_{x\to 1} \frac{e^{x-1}-1}{(x-1)^3}$ $\left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 1} \frac{e^{x-1}}{3(x-1)^2} = \infty$

21. $\lim_{x\to\infty} \dfrac{\ln(x-10)}{\ln(4x+1)} \quad \left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{\left[\dfrac{1}{x-10}\right]}{\left[\dfrac{4}{4x+1}\right]}$

$= \lim_{x\to\infty} \dfrac{4x+1}{4x-40} \quad \left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{4}{4} = \dfrac{4}{4} = 1$

22. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\ln(x+1)} \quad \left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 0^+} \dfrac{\left[\dfrac{1}{2\sqrt{x}}\right]}{\left[\dfrac{1}{x+1}\right]}$

$= \lim_{x\to 0^+} \dfrac{x+1}{2\sqrt{x}} = \infty$

23. $\lim_{x\to\infty} \dfrac{e^{4x}}{e^{3x}+x} \quad \left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{4e^{4x}}{3e^{3x}+1} \quad \left(\text{type } \frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \dfrac{16e^{4x}}{9e^{3x}} = \lim_{x\to\infty} \dfrac{16e^{x}}{9} = \infty$

24. $\lim_{x\to 0} \dfrac{e^{2x}-1}{e^{5x}-1} \quad \left(\text{type } \frac{0}{0}\right)$

$= \lim_{x\to 0} \dfrac{2e^{2x}}{5e^{5x}} = \lim_{x\to 0} \dfrac{2}{5e^{3x}} = \dfrac{2}{5}$