Ordinary Simple Annuities

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Annuities:

- An *annuity* is a series of payments made at periodic intervals. The *term of an annuity* is the length of time between the first and last payment. The size of each payment is the *periodic rent*.

- Types of annuities:

  – *Annuities certain* are annuities where the first and last payment are known.
  Example: payment of loans, mortgage payments, interest payments, lease payments.

  – *Contingent annuities* are annuities where the first and/or last payment are unknown.
  Example: life insurance, pension payments, conditional scholarship.

  – *Perpetuity annuities* are annuities where the term of the annuity is infinite (will never end). This occurs

when the size of the payments is less than or equal to the periodic interest.
Example: scholarship fund.

- Types of date of payments:
  - An *ordinary annuity* has its payment made at the end of each payment period.
  - An *annuity due* has its payment made at the start of each payment period.
  - A *deferred annuity* has its first payment set at a future date.
- Types of payments/compounding:
  - A *simple annuities* compounding coincide with the payment.
  - A *general annuities* compounding is not syncronized with the payments.

Ordinary Simple Annuity: Future Value: Introduction

- Find the future value at the end of the third year of three deposits of $\$1\,000$ each made at the end of three consecutive years, compounded annually at a rate of 10%?

```
Today   1st year  2nd year  3rd year
  |_________|_________|_________|
         $1000     $1000     $1000
            |         |--->  $1000(1+0.1)
            |------------->  $1000(1+0.1)^2
                             ==============
                             $3310
```

Ordinary Simple Annuity: Future Value

- An *ordinary simple annuity* is a geometric progression, its future value is given by the following formula:

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

where FV_n is the future value after n payments, PMT is the size of the equal payments, i is the periodic rate of interest.

Ordinary Simple Annuity: Future Value: Example

- Find the future value of $\$2$ deposits made every day for 3 years, compounded daily at a rate of 4.25%.

$$
\begin{aligned}
i &= \frac{j}{m} = \frac{4.25\%}{365} = 0.000116438 \\
n &= mt = 3(365) = 1095
\end{aligned}
$$

$$
\begin{aligned}
FV_n &= PMT\left[\frac{(1+i)^n - 1}{i}\right] \\
FV_{1095} &= 2\left[\frac{(1+0.000116438)^{1095} - 1}{0.000116438}\right] \\
&= \$2335.60
\end{aligned}
$$

Ordinary Simple Annuity: Future Value: Example

- The IWW has been depositing money into a strike fund. They have made a deposit of $\$500$ each month for the past 5 years into a savings account, compounded monthly at a rate of 5% p.a. They then transfer the fund into a new account where they let the fund mature for 3 years compounded daily at a rate of 4.5%. What is the balance of the strike fund at maturity?

$$
\begin{aligned}
i &= \frac{j}{m} = \frac{5\%}{12} = 0.004166666 \\
n &= mt = 12(5) = 60
\end{aligned}
$$

$$
\begin{aligned}
FV_n &= PMT\left[\frac{(1+i)^n - 1}{i}\right] \\
FV_{60} &= 500\left[\frac{(1+0.004166666)^{60} - 1}{0.004166666}\right] \\
&= \$34003.05
\end{aligned}
$$

$$
\begin{aligned}
i &= \frac{j}{m} = \frac{4.5\%}{365} = 0.000123287 \\
n &= mt = 3(365) = 1095
\end{aligned}
$$

$$
\begin{aligned}
FV &= PV(1+i)^n \\
&= 34003.05(1+0.000123287)^{1095} \\
&= \$38917.42
\end{aligned}
$$

Ordinary Simple Annuity: Present Value: Introduction

- Find the present value of three deposits of $\$1\,000$ each made at the end of three consecutive years, compounded annually at a rate of 10%?

```
Today   1st year  2nd year  3rd year
  |_________|_________|_________|
         $1000     $1000     $1000
$1000(1+0.1)^-1       |         |
$1000(1+0.1)^-2   <---|         |
$1000(1+0.1)^-3   <-------------|
===============
$2486.85
```

Ordinary Simple Annuity: Present Value

- An *ordinary simple annuity* is a geometric progression, its present value is given by the following formula:

  $$PV_n = PMT\left[\frac{1-(1+i)^{-n}}{i}\right]$$

  where PV_n is the present value of n payments, PMT is the size of the equal payments, i is the periodic rate of interest.

Ordinary Simple Annuity: Present Value: Example

- A local of the IWW is planning a strike. They expect the cost of paying the members to be $\$3\,200$ per month and believe the strike will last 12 months. If the current strike fund is at $\$38\,917.42$ and the fund was transferred into an account at a rate of 5% compounded monthly, does the local of the IWW have enough fund?

$$\begin{aligned} i &= \frac{j}{m} = \frac{5\%}{12} = 0.004166666 \\ n &= mt = 12(1) = 12 \end{aligned}$$

$$\begin{aligned} PV_n &= PMT\left[\frac{1-(1+i)^{-n}}{i}\right] \\ PV_{12} &= 3200\left[\frac{1-(1+0.004166666)^{-12}}{0.004166666}\right] \\ &= \$37379.92 \end{aligned}$$

The present value is the required amount needed to strike, it is less than the current strike fund. Therefore they have enough funds to strike.

Ordinary Simple Annuity: Present Value: Application

- If a property is bought on a loan an initial payment called a *down payment* is usually paid complementary to periodic payments. The amount of the loan is the present value of the periodic payments.

- The *cash value* is the value of a property at the date of purchase.

  $$\text{Cash Value} = \text{Down Payment} + \text{PV of the Periodic PMT}$$

Ordinary Simple Annuity: Present Value: Example

- A property was purchased with a down payment of $\$20\,000$ and a payment every month of $\$1\ 500$ for 15 years. If interest is 9% compounded montly, what is the cash value of the property?

$$\begin{aligned} i &= \frac{j}{m} = \frac{9\%}{12} = 0.0075 \\ n &= mt = 12(15) = 180 \end{aligned}$$

$$\begin{aligned} PV_n &= PMT\left[\frac{1-(1+i)^{-n}}{i}\right] \\ PV_{180} &= 1500\left[\frac{1-(1+0.0075)^{-180}}{0.0075}\right] \\ &= \$147890.11 \end{aligned}$$

$$\begin{aligned} \text{Cash Value} &= \text{Down Payment} + \text{PV of the Periodic PMT} \\ &= 20000 + 147890.11 \\ &= \$167890.11 \end{aligned}$$

Ordinary Simple Annuity: Finding the PMT for FV_n:

- *Example:* What deposit made at the end of each six months will accumulate to $\$100\,000$ in 20 years at 2.75% compounded semi-annually?

$$\begin{aligned} i &= \frac{j}{m} = \frac{2.75\%}{2} = 0.01375 \\ n &= mt = 2(20) = 40 \end{aligned}$$

$$\begin{aligned} FV_n &= PMT\left[\frac{(1+i)^n - 1}{i}\right] \\ 100000 &= PMT\left[\frac{(1+0.01375)^{40} - 1}{0.01375}\right] \\ 100000 &= PMT(52.85605608) \\ \$1891.93 &= PMT \end{aligned}$$

Ordinary Simple Annuity: Finding the PMT for PV_n:

- **Example:** What deposit made at the end of each quarter will pay a loan of $\$11\ 000$ in 3 years at 3.5% compounded quartely?

$$\begin{aligned} i &= \frac{j}{m} = \frac{3.5\%}{4} = 0.00875 \\ n &= mt = 3(4) = 12 \end{aligned}$$

$$\begin{aligned} PV_n &= PMT\left[\frac{1-(1+i)^{-n}}{i}\right] \\ 11000 &= PMT\left[\frac{1-(1+0.00875)^{-12}}{0.00875}\right] \\ 11000 &= PMT(11.34447929) \\ \$969.63 &= PMT \end{aligned}$$

Ordinary Simple Annuity: Finding the n for FV_n:

- **Example:** How long will it take for $\$2$ deposited every day to accumulate to $\$608.06$ at 3.25% compounded daily?

$$\begin{aligned}
i &= \frac{j}{m} = \frac{3.25\%}{365} = 0.000089041 \\
FV_n &= PMT\left[\frac{(1+i)^n - 1}{i}\right] \\
608.06 &= 2\left[\frac{(1+0.000089041)^n - 1}{0.000089041}\right] \\
304.03 &= \left[\frac{1.000089041^n - 1}{0.000089041}\right] \\
0.027071135 &= 1.000089041^n - 1 \\
1.027071135 &= 1.000089041^n \\
\ln 1.027071135 &= \ln 1.000089041^n \\
\ln 1.027071135 &= n \ln 1.000089041 \\
n &= 300 \text{ days}
\end{aligned}$$

Ordinary Simple Annuity: Finding the n for PV_n:

- *Example:* How many payments of $\$545$ made every three months are required to repay a loan of $\$9\,130.32$ at 7% compounded quartely?

$$
\begin{aligned}
i &= \frac{j}{m} = \frac{7\%}{4} = 0.0175 \\
PV_n &= PMT\left[\frac{1-(1+i)^{-n}}{i}\right] \\
9130.32 &= 545\left[\frac{1-(1+0.0175)^{-n}}{0.0175}\right] \\
16.7528813 &= \left[\frac{1-1.0175^{-n}}{0.0175}\right] \\
0.293175422 &= 1-1.0175^{-n} \\
1.0175^{-n} &= 0.706824577 \\
\ln 1.0175^{-n} &= \ln 0.706824577 \\
-n\ln 1.0175 &= \ln 0.706824577 \\
n &= 20 \text{ payments}
\end{aligned}
$$