Test 2

Question 1. *(5 marks)* Decide whether or not each of the following matrices are invertible. Explain why or why not. For each matrix that is invertible find the inverse.

$$A=\begin{bmatrix}1&1&7\\0&0&-2\\0&0&5\end{bmatrix}\quad B=\begin{bmatrix}1&6&3\\3&7&9\\2&1&6\end{bmatrix}\quad C=\begin{bmatrix}1&0&0\\0&3&0\\0&0&-3\end{bmatrix}$$

Question 2. *(9 marks)*
a) If A is symmetric show that $4A^2-A^TA$ is symmetric.
b) Find the values of a, b and c for which the following matrix B is symmetric:

$$B=\begin{bmatrix}-2&a&b\\6&9&c^2+4c\\-2&-4&0\end{bmatrix}$$

Question 3. *(8 marks)* Find $\det A$ using cofactor expansion:

$$A=\begin{bmatrix}1&0&3&-1\\2&0&2&0\\8&3&7&0\\3&0&-2&2\end{bmatrix}$$

Question 4. *(10 marks)* Solve the following system using Cramer's rule:

$$\begin{array}{rcrcrcl}3x_1&-&x_2&+&x_3&=&2\\2x_1&+&x_2&-&x_3&=&1\\&&5x_2&-&3x_3&=&0\end{array}$$

Question 5. *(8 marks)* Find the $\det A$ by first reducing A to **row-echelon form**.

$$A=\begin{bmatrix}2&0&4\\1&-6&2\\3&9&12\end{bmatrix}$$

Question 6. *(8 marks)* Given that A, B, C are 2×2 matrices and that $\det A=3$, $\det B=5$, and $\det(4C)=9$, find:
(a) $\det(AB^T)$
(b) $\det\left((4A)^{-1}(3B)\right)$
(c) $\det(C)$

Question 7. *(9 marks)* Given vectors $\vec{\mathbf{v}}=(4,1,2)$, $\vec{\mathbf{u}}=(-1,3,2)$, $\vec{\mathbf{w}}=(-2,0,-2)$ find:
(a) $4\vec{\mathbf{v}}-2\vec{\mathbf{u}}$
(b) $-5(\vec{\mathbf{u}}+3\vec{\mathbf{w}})$
(c) $||\vec{\mathbf{u}}-\vec{\mathbf{v}}||-2||\vec{\mathbf{w}}||$

Question 8. *(7 marks)* Find the TERMINAL point of a vector $\vec{\mathbf{u}}$ with initial point $P_1(-1,-2,5)$ such that:
(a) $\vec{\mathbf{u}}$ has the same direction as $\vec{\mathbf{v}}=(0,4,3)$
(b) $\vec{\mathbf{u}}$ is oppositely directed to $\vec{\mathbf{v}}=(0,4,4)$.
(c) Find the distance between $P_1(-1,-2,5)$ and $P_3(2,0,1)$

Bonus Question *(2 marks)* Using the right-handed coordinate system draw the pyramid whose base has the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(2,2,0)$ and whose top point is $(1,1,2)$.

TEST 2 SOLUTIONS

LINEAR ALGEBRA 201-105-DW 04

1) A IS NOT INVERTABLE SINCE $\det A = 0$

B IS NOT INVERTABLE SINCE $C_3 = 3C_1$ $(\det B = 0)$

C IS INVERTABLE SINCE $\det C = -9 \neq 0$

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

2) a) WE WANT TO SHOW $(4A^2 - A^TA)^T = 4A^2 - A^TA$

$$(4A^2 - A^TA)^T = (4A^2)^T - (A^TA)^T = 4(A^2)^T - (A^T)(A^T)^T$$

$$= 4(A^T)^2 - A^TA = 4A^2 - A^TA$$

AS REQUIRED $\therefore$ $4A^2 - A^TA$ IS SYMMETRIC

b) $B = \begin{bmatrix} -2 & a & b \\ 6 & 9 & c^2+4c \\ -2 & -4 & 0 \end{bmatrix}$ IS SYMMETRIC WHEN

$a = 6$, $b = -2$, $c^2 + 4c = -4$

$$\Rightarrow c^2 + 4c + 4 = 0$$

$$(c+2)(c+2) = 0$$

$$\therefore c = -2$$

$$\boxed{a = 6,\ b = -2,\ c = -2}$$

3) $\det A = \det \begin{bmatrix} 1 & 0 & 3 & -1 \\ 2 & 0 & 2 & 0 \\ 8 & 3 & 7 & 0 \\ 3 & 0 & -2 & 2 \end{bmatrix}$

$= 0 + 0 - 3\det \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & 0 \\ 3 & -2 & 2 \end{bmatrix} + 0$

$= -3\left(-(1)\det \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix} + 2\det \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\right)$

$= -3\left(-(-4-6) + 2(2-6)\right) = -3(10-8)$

$= -6$

4) THE SYSTEM CAN BE WRITTEN AS

$$\begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & -1 \\ 0 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

AND SO

$A = \begin{bmatrix} 3 & -1 & 1 \\ 2 & 1 & -1 \\ 0 & 5 & -3 \end{bmatrix}$, $A_1 = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 5 & -3 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

$$A_3 = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 0 & 5 & 0 \end{bmatrix}$$

$$\det A = 3\det\begin{bmatrix}1 & -1\\ 5 & -3\end{bmatrix} - 2\det\begin{bmatrix}-1 & 1\\ 5 & -3\end{bmatrix} + 0$$

$$= 3(-3+5) - 2(3-5) = 6+4 = 10$$

$$\det A_1 = 2\det\begin{bmatrix}1 & -1\\ 5 & -3\end{bmatrix} - (1)\det\begin{bmatrix}-1 & 1\\ 5 & -3\end{bmatrix} + 0$$

$$= 2(2) - (-2) = 6$$

$$\det A_2 = 0 + 0 - 3\det\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix} = -3(3-4) = 3$$

$$\det A_3 = 0 - 5\det\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix} + 0 = -5(3-4) = 5$$

$$\therefore\quad x_1 = \frac{\det A_1}{\det A} = \frac{6}{10} = \frac{3}{5},\quad x_2 = \frac{\det A_2}{\det A} = \frac{3}{10}$$

$$x_3 = \frac{\det A_3}{\det A} = \frac{5}{10} = \frac{1}{2}$$

$$\therefore\quad (x_1, x_2, x_3) = (3/5, 3/10, 1/2)$$

5) $A = \begin{bmatrix} 2 & 0 & 4 \\ 1 & -6 & 2 \\ 3 & 9 & 12 \end{bmatrix} \xrightarrow{R_1 \cdot (\frac{1}{2})} \begin{bmatrix} 1 & 0 & 2 \\ 1 & -6 & 2 \\ 3 & 9 & 12 \end{bmatrix} \xrightarrow[R_3 - 3R_1]{R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -6 & 0 \\ 0 & 9 & 6 \end{bmatrix}$

$\xrightarrow{R_2 \cdot (\frac{-1}{6})} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 9 & 6 \end{bmatrix} \xrightarrow{R_3 - 9R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_3 \cdot (\frac{1}{6})} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B$

OPERATIONS THAT AFFECT THE DETERMINANT:

$R_1 \cdot (\frac{1}{2})$, $R_2 (-\frac{1}{6})$, $R_3 \cdot (\frac{1}{6})$

$\therefore (\frac{1}{2})(-\frac{1}{6})(\frac{1}{6}) \det A = \det B = 1$

$\therefore \det A = -72$

6) a) $\det(AB^T) = \det A \det(B^T) = \det A \det B$

$= (3)(5) = 15$

b) $\det((4A)^{-1}(3B)) = \det((4A)^{-1}) \det(3B)$

$= \frac{1}{\det(4A)} \cdot \det(3B) = \frac{1}{4^2 \det A} \cdot 3^2 \det B$

$= \frac{1}{(16)(3)} \cdot 9(5) = \frac{15}{16}$

c) $\det(4C) = 9$

$4^2 \det C = 9$

$\det C = \frac{9}{16}$

7a) $4\vec{v} - 2\vec{u} = 4(4,1,2) - 2(-1,3,2)$

$= (16,4,8) - (-2,6,4) = (18,-2,4)$

b) $-5(\vec{u} + 3\vec{w}) = -5\left((-1,3,2) + 3(-2,0,-2)\right)$

$= -5(-7,3,-4) = (35,-15,20)$

c) $\|\vec{u} - \vec{v}\| - 2\|\vec{w}\| = \|(-1,3,2) - (4,1,2)\| - 2\|(-2,0,-2)\|$

$= \|(-5,2,0)\| - 2\sqrt{(-2)^2 + 0^2 + (-2)^2}$

$= \sqrt{(-5)^2 + (2)^2 + (0)^2} - 2\sqrt{4+4} = \sqrt{29} - 2\sqrt{8}$

$= \sqrt{29} - 4\sqrt{2}$

8) TERMINAL POINT: $P_2 = (x, y, z)$

a) $(x-(-1),\ y-(-2),\ z-(5)) = (0,4,3)$

$\Rightarrow x = -1,\ y = 2,\ z = 8$

$\therefore P_2 = (-1, 2, 8)$

b) $-(0,4,4) = (0,-4,-4)$ IS OPPOSITELY DIRECTED TO $(0,4,4)$

$\therefore (x-(-1),\ y-(-2),\ z-(5)) = (0,-4,-4)$

$\Rightarrow x = -1,\ y = -6,\ z = 1$

$\therefore P_2 = (-1,-6,1)$

c) $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

$= \sqrt{(2-(-1))^2 + (0-(-2))^2 + (1-5)^2}$

$= \sqrt{3^2 + 2^2 + (-4)^2}$

$= \sqrt{29}$

BONUS QUESTION: