Dawson College: Linear Algebra: 201-105-DW Section 05 March 30, 2009

Test 2

Question 1. *(5 marks)* Find A^3, A^{-2}, and A^k by inspection:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Question 2. *(8 marks)* Find the values of a, b and c such that the following matrix is symmetric.

$$\begin{bmatrix} 7 & a+b & 3a-b \\ -2 & 0 & 3b \\ 6 & 2b-c+1 & 2 \end{bmatrix}$$

Question 3. *(10 marks)* Solve the following system using Cramer's Rule:

$$\begin{aligned} -x_1 + 2x_2 - 3x_3 &= 1 \\ 2x_1 + x_3 &= 0 \\ 3x_1 - 4x_2 + 4x_3 &= 2 \end{aligned}$$

Question 4. *(8 marks)* Find $\det A$ by first reducing A to **row-echelon form**:

$$A = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 6 & 3 \\ -3 & -3 & 12 \end{bmatrix}$$

Question 5. *(8 marks)* Given that A, B, C are 3×3 matrices and that $\det A = -4$, $\det B = 2$, and $\det(5C) = 7$, find:

(a) $\det(AB^{-1})$

(b) $\det\left((3A)^{-1}(2B)\right)$

(c) $\det(C)$

Question 6. *(9 marks)* For what values of k is the following matrix not invertible?

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & k+1 \\ k & 0 & 3-k \end{bmatrix}$$

Question 7. *(9 marks)* Given vectors $\vec{\mathbf{v}} = (1,3,2)$, $\vec{\mathbf{u}} = (-2,2,4)$, $\vec{\mathbf{w}} = (-1,0,-1)$ find:

(a) $3\vec{\mathbf{v}} - 2\vec{\mathbf{u}}$

(b) $-4(2\vec{\mathbf{u}} + \vec{\mathbf{w}})$

(c) $||\vec{\mathbf{u}}|| - ||\vec{\mathbf{u}} + 3\vec{\mathbf{w}}||$

Question 8. *(7 marks)* Find the terminal point of a vector $\vec{\mathbf{u}}$ with initial point $P_1(1,2,2)$ such that:

(a) $\vec{\mathbf{u}}$ has the same direction as $\vec{\mathbf{v}} = (2,3,0)$

(b) $\vec{\mathbf{u}}$ is oppositely directed to $\vec{\mathbf{v}} = (2,3,0)$.

(c) Find the distance between $P_1(1,2,2)$ and $P_3(-2,0,5)$

Bonus Question *(2 marks)* Using the right-handed coordinate system draw the pyramid whose base has the points $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(2,2,0)$ and whose top point is $(1,1,2)$.

TEST 2 SOLUTIONS

LINEAR ALGEBRA 201-105-DW 05

1) $A^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{8}{27} & 0 & 0 \\ 0 & 0 & 64 & 0 \\ 0 & 0 & 0 & -27 \end{bmatrix}$, $A^{-2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{9}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{bmatrix}$

$$A^K = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2^K}{3^K} & 0 & 0 \\ 0 & 0 & 4^K & 0 \\ 0 & 0 & 0 & (-3)^K \end{bmatrix}$$

2) THE MATRIX IS SYMMETRIC IF AND ONLY IF

$a+b=-2$

$3a-b=6$

$2b-c+1=3b \Rightarrow b+c=1$

SOLVING THIS SYSTEM:

$$\begin{bmatrix} 1 & 1 & 0 & -2 \\ 3 & -1 & 0 & 6 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-3R_1} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & -4 & 0 & 12 \\ 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2\cdot(-\frac{1}{4})} \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3+R_2]{R_1-R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

i.e. $\boxed{a=1, \quad b=-3, \quad c=4}$

3) THE MATRIX EQUATION FOR THE SYSTEM IS

$$\begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \text{ AND SO } A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{bmatrix}$$

$$\det A_1 = 0 + 0 - (1)\det \begin{bmatrix} 1 & 2 \\ 2 & -4 \end{bmatrix} = -(-4-4) = 8$$

$$\det A_2 = -2\det \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} + 0 - (1)\det \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} =$$

$$= -2(4+6) - (-2-3) = -20 + 5 = -15$$

$$\det A_3 = -2\det \begin{bmatrix} 2 & 1 \\ -4 & 2 \end{bmatrix} + 0 - 0 = -2(4+4) = -16$$

$$\det A = -2\det \begin{bmatrix} 2 & -3 \\ -4 & 4 \end{bmatrix} + 0 - (1)\det \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

$$= -2(8-12) - (4-6) = 8 + 2 = 10$$

$$X_1 = \frac{\det A_1}{\det A} = \frac{8}{10} = \frac{4}{5}, \quad X_2 = \frac{\det A_2}{\det A} = \frac{-15}{10} = \frac{-3}{2}$$

$$X_3 = \frac{\det A_3}{\det A} = \frac{-16}{10} = \frac{-8}{5}$$

$$\therefore (X_1, X_2, X_3) = \left(\tfrac{4}{5}, \tfrac{-3}{2}, \tfrac{-8}{5}\right)$$

4) $A = \begin{bmatrix} 0 & 1 & 3 \\ 3 & 6 & 3 \\ -3 & -3 & 12 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 6 & 3 \\ 0 & 1 & 3 \\ -3 & -3 & 12 \end{bmatrix} \xrightarrow{R_3 + R_1} \begin{bmatrix} 3 & 6 & 3 \\ 0 & 1 & 3 \\ 0 & 3 & 15 \end{bmatrix}$

$\xrightarrow{R_1 \cdot (1/3)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 15 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 6 \end{bmatrix} \xrightarrow{R_3 \cdot (1/6)} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = B$

OPERATIONS THAT AFFECT THE DETERMINANT

$R_1 \leftrightarrow R_2$, $R_1 \cdot (1/3)$, $R_3 \cdot (1/6)$

$$-\left(\tfrac{1}{3}\right)\left(1/6\right) \det A = \det B$$

$$-\tfrac{1}{18} \det A = 1$$

$$\det A = -18$$

5) a) $\det(AB^{-1}) = \det(A)\det(B^{-1})$

$$= \det A \cdot \frac{1}{\det B} = -4\left(\tfrac{1}{2}\right) = -2$$

b) $\det((3A)^{-1}(2B)) = \det((3A)^{-1})\det(2B)$

$$= \frac{1}{\det(3A)} \cdot \det(2B)$$

$$= \frac{1}{3^3 \det A} \cdot 2^3 \det B$$

$$= \frac{1}{27(-4)} \cdot 8(2) = -\frac{4}{27}$$

c) $7 = \det(5C) = 5^3 \det(C) = 125 \det(C)$

$$\therefore \det(C) = \frac{7}{125}$$

6) THE MATRIX A IS NOT INVERTABLE IF AND ONLY IF $\det A = 0$.

$$\det A = \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & k+1 \\ k & 0 & 3-k \end{bmatrix} = 0 - (1)\det \begin{bmatrix} 1 & k+1 \\ k & 3-k \end{bmatrix} + 0$$

$$= -(3-k-(k)(k+1)) = -(3-k-k^2-k)$$

$$= k^2+2k-3 = 0$$

$$\Rightarrow (k+3)(k-1) = 0$$

$$k = -3, \underline{1}$$

7) a) $3\vec{v} - 2\vec{u} = 3(1,3,2) - 2(-2,2,4) =$

$= (3,9,6) - (-4,4,8) = (7,5,-2)$

b) $-4(2\vec{u} + \vec{\omega}) = -4\left(2(-2,2,4) + (-1,0,-1)\right)$

$= -4((-4,4,8) + (-1,0,-1)) = -4(-5,4,7)$

$= (20,-16,-28)$

c) $\|\vec{u}\| - \|\vec{u} + 3\vec{\omega}\| = \|(-2,2,4)\| - \|(-2,2,4) + 3(-1,0,-1)\|$

$= \|(-2,2,4)\| - \|(-5,2,1)\| = \sqrt{(-2)^2+(2)^2+(4)^2} - \sqrt{(-5)^2+(2)^2+(1)^2}$

$= \sqrt{4+4+16} - \sqrt{25+4+1} = \sqrt{24} - \sqrt{30} = 2\sqrt{6} - \sqrt{30}$

5) a) LET THE TERMINAL POINT BE $P_2(x, y, z)$

THEN $(2, 3, 0) = (x-1, y-2, z-2)$

$\Rightarrow x = 3, \quad y = 5, \quad z = 2$

$\therefore$ THE TERMINAL POINT IS $P_2(3, 5, 2)$

b) TERMINAL POINT $= P_2(x, y, z)$

$(-2, -3, 0)$ IS OPPOSITELY DIRECTED TO $\vec{v}$

$\therefore \quad (-2, -3, 0) = (x-1, y-2, z-2)$

$\therefore \quad x = -1, \quad y = -1, \quad z = 2$

$\therefore$ THE TERMINAL POINT IS $P_2(-1, -1, 2)$

c) THE DISTANCE BETWEEN P_1 AND P_2 IS

$$d = \sqrt{(-2-1)^2 + (0-2)^2 + (5-2)^2} = \sqrt{(-3)^2 + (-2)^2 + (3)^2}$$

$$= \sqrt{9 + 4 + 9} = \sqrt{22}$$

BONUS QUESTION