Test 3

No books or cell phones are allowed. You may use a nonprogrammable scientific calculator without graphing capabilities. Show all of your work and use correct notation.

Question 1. *(7 marks)* Find each limit:

(b) $\lim_{x\to 0} \frac{\sin x^2 + 2x}{e^x - 1}$ **(b)** $\lim_{x\to\infty} \frac{2x^{3/2}}{\ln x}$ **(c)** $\lim_{x\to 0} \frac{e^x - x - 1}{3x^2}$

Question 2. *(4 marks)* Evaluate the improper integral:

$$\int_4^{\infty} \frac{5}{x^{3/2}} dx$$

Question 3. *(4 marks)* Find the area under the curve $y = e^{x/2}$ for $x \leq 2$.

Question 4. *(4 marks)* Verify that $y = xe^{3x}$ a solution of the differential equation

$$y'' - 6y' + 6y = -3xe^{3x}$$

Question 5. *(6 marks)* Use separation of variables to solve the following first order differential equation

$$y' = \frac{3xy^2}{(x^2+1)^3}$$

subject to the initial condition $y(0) = 1$.

Question 6. *(5 marks)* Find the fourth Taylor polynomial for $y = \sqrt{6x-5}$ at $x = 1$.

Question 7. *(7 marks)* Determine whether the following sequences converge. If it converges determine what it converges to.

(a) $a_n = \frac{2n^3 - 3n - 6}{2n^2 - 7}$ **(b)** $a_n = \frac{\sqrt[5]{n} - 1}{\sqrt[5]{n} + 2}$ **(d)** $a_n = \frac{6n}{n!}$

Question 8. *(7 marks)* Determine whether or not the following series converge or diverge. If it converges find the sum.

(a) $\sum_{n=1}^{\infty} \left(\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} \right)$ **(b)** $\sum_{n=0}^{\infty} 7 \left(\frac{5}{3} \right)^n$ **(c)** $\sum_{n=0}^{\infty} \frac{3^n - 4 \cdot 2^n}{7^n}$

Bonus. *(1 mark)* A Taylor series is an infinite Taylor polynomial, that is, a Taylor polynomial with infinite number of terms. Find the Taylor series for e^x at $x = 0$. (It turns out that this Taylor series is equal to e^x !)

TEST 3 SOLUTIONS

1 a) $$\lim_{x\to 0} \frac{\sin x^2 + 2x}{e^x - 1} \overset{H}{=} \lim_{x\to 0} \frac{2x\cdot\cos x^2 + 2}{e^x}$$

FORM $\frac{0}{0}$

$$= \frac{2(0)\cos(0)+2}{e^0} = \frac{0(1)+2}{1}$$

$$= 2$$

b) $$\lim_{x\to\infty} \frac{2x^{3/2}}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{2\cdot\frac{3}{2}x^{1/2}}{\frac{1}{x}} = \lim_{x\to\infty} 3x^{1/2}\cdot x$$

FORM $\frac{\infty}{\infty}$

$$= \lim_{x\to\infty} 3x^{3/2} = \infty$$ DIVERGES

c) $$\lim_{x\to 0} \frac{e^x - x - 1}{3x^2} \overset{H}{=} \lim_{x\to 0} \frac{e^x - 1}{6x}$$

FORM $\frac{0}{0}$

FORM $\frac{0}{0}$

$$\overset{H}{=} \lim_{x\to 0} \frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}$$

2)

$$\int_4^\infty \frac{5}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_4^b \frac{5}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_4^b 5x^{-3/2}\,dx$$

$$= \lim_{b\to\infty}\left[5\,\frac{x^{-1/2}}{-\frac{1}{2}}\right]_4^b = \lim_{b\to\infty}\left[-10\,b^{-1/2} + 10\,4^{-1/2}\right]$$

$$= \lim_{b\to\infty}\left[\frac{-10}{\sqrt{b}} + \frac{10}{\sqrt{4}}\right] = 0 + \frac{10}{2} = 5$$

3) AREA $= \displaystyle\int_{-\infty}^{2} e^{x/2}\,dx = \lim_{a\to-\infty}\int_a^2 e^{x/2}\,dx$

$$= \lim_{a\to-\infty}\left[2e^{x/2}\right]_a^2 = \lim_{a\to-\infty}\left[2e^{2/2} - 2e^{a/2}\right]$$

$$= 2e^1 - 0 = 2e \text{ units}^2$$

4) $y = xe^{3x}$

$y' = e^{3x} + 3xe^{3x}$

$y'' = 3e^{3x} + 3e^{3x} + 9xe^{3x} = 6e^{3x} + 9xe^{3x}$

LHS $= y'' - 6y' + 6y =$

$= (6e^{3x} + 9xe^{3x}) - 6(e^{3x} + 3xe^{3x}) + 6(xe^{3x})$

$= 6e^{3x} + 9xe^{3x} - 6e^{3x} - 18xe^{3x} + 6xe^{3x}$

$= -3xe^{3x} = $ RHS $\quad\therefore$ y IS A SOLUTION OF THE DIFFERENTIAL EQUATION

5) $y' = \dfrac{3xy^2}{(x^2+1)^3} \Rightarrow \dfrac{dy}{dx} = \dfrac{3x}{(x^2+1)^3} \cdot y^2$

$\Rightarrow \dfrac{1}{y^2}\,dy = \dfrac{3x}{(x^2+1)^3}\,dx \Rightarrow \displaystyle\int \frac{1}{y^2}\,dy = \int \frac{3x}{(x^2+1)^3}\,dx$

$LHS = \displaystyle\int \frac{1}{y^2}\,dy = \int y^{-2}\,dy = \frac{y^{-1}}{-1} + C_1 = -\frac{1}{y} + C_1$

$RHS = \displaystyle\int \frac{3x}{(x^2+1)^3}\,dx$

$$\boxed{\begin{aligned} \text{LET} \quad u &= x^2+1 \\ du &= 2x\,dx \\ dx &= \frac{du}{2x} \end{aligned}}$$

$= \displaystyle\int \frac{3\cancel{x}}{u^3}\,\frac{du}{2\cancel{x}}$

$= \dfrac{3}{2}\displaystyle\int u^{-3}\,du = \frac{3}{2}\,\frac{u^{-2}}{-2} + C_2 = -\frac{3}{4}(x^2+1)^{-2} + C_2$

$= \dfrac{-3}{4(x^2+1)^2} + C_2$

$\therefore \quad -\dfrac{1}{y} + C_1 = \dfrac{-3}{4(x^2+1)^2} + C_2$

$-\dfrac{1}{y} = \dfrac{-3}{4(x^2+1)^2} + C \qquad (C = C_2 - C_1)$

$y(0) = 1$; $\quad -\dfrac{1}{1} = \dfrac{-3}{4(0^2+1)^2} + C \Rightarrow -1 = -\dfrac{3}{4} + C$

$\Rightarrow -1 + \dfrac{3}{4} = C \Rightarrow C = -\dfrac{1}{4}$

$$\therefore \ -\frac{1}{y} = \frac{-3}{4(x^2+1)} - \frac{1}{4}$$

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6) $f(x) = \sqrt{6x-5} = (6x-5)^{1/2}$

$$f'(x) = \tfrac{1}{2}(6x-5)^{-1/2} \cdot (6) = 3(6x-5)^{-1/2}$$

$$f''(x) = 3\left(-\frac{1}{2}\right)(6x-5)^{-3/2}(6) = -9(6x-5)^{-3/2}$$

$$f'''(x) = -9\left(-\frac{3}{2}\right)(6x-5)^{-5/2}(6) = 81(6x-5)^{-5/2}$$

$$f^{(4)}(x) = 81\left(\frac{-5}{2}\right)((6x-5)^{-7/2}(6) = -1215(6x-5)^{-7/5}$$

$\therefore f(1) = 1, \quad f'(1) = 3, \quad f''(1) = -9, \quad f'''(1) = 81$

$f^{(4)}(1) = -1215$

$$\therefore T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(x)}{3!}(x-1)^3 + \frac{f^{(4)}}{4!}(x-1)^4$$

$$= 1 + 3(x-1) - \frac{9}{2}(x-1)^2 + \frac{81}{6}(x-1)^3 - \frac{1215}{24}(x-1)^4$$

$$= 1 + 3(x-1) - \frac{9}{2}(x-1)^2 + \frac{27}{2}(x-1)^3 - \frac{405}{8}(x-1)^4$$

7) a) $\lim_{n\to\infty} \frac{2n^3-3n-6}{2n^2-7} = \lim_{n\to\infty} \frac{2n - 3/n - 6/n^2}{2 - 7/n^2} = \infty$

DIVERGES.

b) $\lim_{n\to\infty} \frac{\sqrt[5]{n}-1}{\sqrt[5]{n}+2} = \lim_{n\to\infty} \frac{n^{1/5}-1}{n^{1/5}+2} = \lim_{n\to\infty} \frac{1-1/n^5}{1-2/n^5}$

$= \frac{1-0}{1-0} = 1$ CONVERGES

c) $\lim_{n\to\infty} \frac{6n}{n!} = \lim_{n\to\infty} \frac{6\cancel{n}}{1\cdot 2\cdot 3\cdot\ldots\cdot(n-1)\cancel{n}} = \lim_{n\to\infty} \frac{6}{(n-1)!}$

$= 0$

8) a) $S_1 = \frac{1}{2^2} - \frac{1}{3^2}$, $S_2 = \left(\frac{1}{2^2} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \frac{1}{4^2}\right)$

$= \frac{1}{2^2} - \frac{1}{4^2}$

$S_3 = \left(\frac{1}{2^2} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}\right) + \left(\cancel{\frac{1}{4^2}} - \frac{1}{5^2}\right) = \frac{1}{2^2} - \frac{1}{5^2}$

$S_N = \left(\frac{1}{2^2} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}\right) + \left(\cancel{\frac{1}{4^2}} - \cancel{\frac{1}{5^2}}\right) + \left(\cancel{\frac{1}{5^2}} - \cancel{\frac{1}{6^2}}\right)$

$+ \ldots + \left(\cancel{\frac{1}{N^2}} - \cancel{\frac{1}{(N+1)^2}}\right) + \left(\cancel{\frac{1}{(N+1)^2}} - \frac{1}{(N+2)^2}\right)$

$= \frac{1}{2^2} - \frac{1}{(N+2)^2}$

$\sum_{n=1}^{\infty} \left(\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2}\right) = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2^2} - \frac{1}{(N+2)^2}\right)$

$= \frac{1}{2^2} - 0 = \frac{1}{4}$ CONVERGES

b) GEOMETRIC SERIES WITH $a = 7$

$|r| = \frac{5}{3} > 1 \quad \therefore$ THE SERIES $\sum_{n=0}^{\infty} 7\left(\frac{5}{3}\right)^n$ DIVERGES

c) $$\sum_{n=0}^{\infty} \frac{3^n - 4\cdot 2^n}{7^n} = \sum_{n=0}^{\infty} \frac{3^n}{7^n} - 4\sum_{n=0}^{\infty} \frac{2^n}{7^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{3}{7}\right)^n - 4\sum_{n=0}^{\infty}\left(\frac{2}{7}\right)^n$$ (GEOMETRIC SERIES $|r| = \frac{3}{7} < 1$, $|r| = \frac{2}{7} < 1$)

$$= \frac{1}{1-\frac{3}{7}} - 4\left(\frac{1}{1-\frac{2}{7}}\right) = \frac{1}{\frac{4}{7}} - 4\left(\frac{1}{\frac{5}{7}}\right)$$

$$= \frac{7}{4} - 4\left(\frac{7}{5}\right) = \frac{7}{4} - \frac{28}{5} = \frac{35}{20} - \frac{192}{20} = \frac{-157}{20}$$