Dawson College: Linear Algebra: 201-105-DW Section 04 May 6, 2009

Test 3

Question 1. *(5 marks)* Given $\vec{\mathbf{u}} = (5,-1,3)$, $\vec{\mathbf{v}} = (2,3,1)$ and $\vec{\mathbf{w}} = (-2,0,3)$ find the following (if possible):

(a) $(\vec{\mathbf{u}} \cdot \vec{\mathbf{v}})\vec{\mathbf{w}}$ **(b)** $\vec{\mathbf{w}} - (\vec{\mathbf{u}} \times \vec{\mathbf{v}})$ **(c)** $\vec{\mathbf{v}} \times (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})$

Question 2. *(4 marks)*
(a) The projection of $\vec{\mathbf{u}} = (4,1,-5)$ onto $\vec{\mathbf{a}} = (4,-2,2)$.
(b) Write $\vec{\mathbf{u}}$ as the sum of 2 vectors, one parallel to $\vec{\mathbf{a}}$ and one perpendicular to $\vec{\mathbf{a}}$ (indicate which is which).

Question 3. *(6 marks)* Let $\vec{\mathbf{u}} = (3,1,-5)$ onto $\vec{\mathbf{v}} = (-1,3,2)$.
(a) Let θ be the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$. Find $\cos\theta$.
(b) Find the area of the parallelogram determined by $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

Question 4. *(4 marks)* Determine whether or not the planes $2x-3y+7z+3=0$ and $-6x+9y-21z-5=0$ are parallel. Explain your answer.

Question 5. *(4 marks)* Find the volume of the parallelepiped determined by $\vec{\mathbf{u}} = (-2,4,3)$, $\vec{\mathbf{v}} = (-3,0,3)$, and $\vec{\mathbf{w}} = (1,5,-1)$.

Question 6. *(4 marks)* Find the equation of the line through the point $P(3,-1,7)$ and perpendicular to the plane $-2x+y+5z-13=0$.

Question 7. *(5 marks)* Find the equation of the plane passing through the points $A(1,3,-4)$, $B(-1,2,3)$ and $C(0,1,-3)$.

Question 8. *(6 marks)*
(a) Find the equation of the line of intersection of the planes $2x-2y+z=1$ and $x+y-z=-2$.
(b) Find the equation of the line that is parallel to both the above planes and passing through the point $P(1,-4,6)$.

Question 9. *(5 marks)* Find the distance between the plane $4x-2y-8z+9=10$ and the point $(1,0,2)$.

Bonus. *(3 marks)* Prove that $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -(\vec{\mathbf{u}} \times \vec{\mathbf{w}}) \cdot \vec{\mathbf{v}}$

SOLUTIONS

1) a) $(\vec{u}\cdot\vec{v})\vec{w} = ((5,-1,3)\cdot(2,3,1))(-2,0,3)$

$= 10(-2,0,3) = (-20,0,30)$

b) $\vec{u}\times\vec{v} = \left(\det\begin{bmatrix}-1 & 3\\ 3 & 1\end{bmatrix}, -\det\begin{bmatrix}5 & 2\\ 3 & 1\end{bmatrix}, \det\begin{bmatrix}5 & 2\\ -1 & 3\end{bmatrix}\right)$

$= (-10, 1, 17)$

$\vec{w} - \vec{u}\times\vec{v} = (-2,0,3) - (-10,1,17) = (8,-1,-14)$

c) NOT POSSIBLE

2a) $\vec{w}_1 = \text{proj}_{\vec{a}}\vec{u} = \dfrac{\vec{u}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a} = \dfrac{(4,1,-5)\cdot(4,-2,2)}{(4,-2,2)\cdot(4,-2,2)}(4,-2,2)$

$= \dfrac{4}{24}(4,-2,2) = \dfrac{1}{6}(4,-2,2) = \left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{1}{3}\right)$

b) $w_2 = \vec{u} - \vec{w}_1 = (4,1,-5) - \left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{1}{3}\right)$

$= \left(\dfrac{10}{3}, \dfrac{4}{3}, \dfrac{-16}{3}\right)$

$\vec{u} = \vec{w}_{\parallel} + \vec{w}_2 = \underbrace{\left(\dfrac{2}{3}, \dfrac{-1}{3}, \dfrac{1}{3}\right)}_{\text{PARALLEL}} + \underbrace{\left(\dfrac{10}{3}, \dfrac{4}{3}, \dfrac{-16}{3}\right)}_{\text{PERPENDICULAR}}$

3) a) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \dfrac{-3+3-10}{\sqrt{(3)^2+(1)^2+(-5)^2}\cdot\sqrt{(-1)^2+(3)^2+(2)^2}}$

$= \dfrac{-10}{\sqrt{35}\sqrt{14}} = \dfrac{-10}{\sqrt{490}} = \dfrac{-10}{7\sqrt{10}}$

b) AREA = $\|\vec{u} \times \vec{v}\|$

$\vec{u} \times \vec{v} = \left(\det\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}, -\det\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}, \det\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\right)$

$\begin{matrix} 4 & -1 \\ 1 & 3 \\ -5 & 2 \end{matrix} = (17, -1, 10)$

$\|\vec{u} \times \vec{v}\| = \sqrt{(17)^2 + (-1)^2 + (10)^2} = \sqrt{390}$

4) THE PLANES ARE PARALLEL IF AND ONLY IF THEIR NORMAL VECTORS ARE PARALLEL

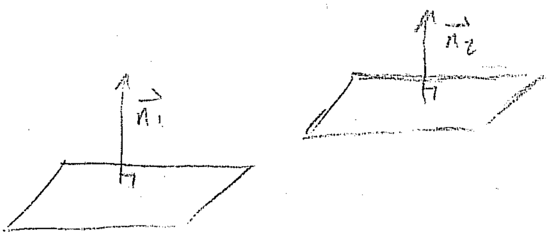
THE NORMALS OF THE PLANES ARE

$\vec{n}_1 = (2, -3, 7)$, $\vec{n}_2 = (-6, 9, -21)$

SINCE $\vec{n}_2 = -3\vec{n}_1$ (SCALAR MULTIPLES)

THE NORMAL VECTORS ARE PARALLEL

$\therefore$ THE PLANES ARE PARALLEL



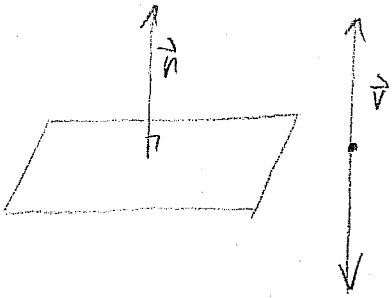
5) VOLUME = $|\vec{u} \cdot (\vec{v} \times \vec{w})| = \left| \det\begin{bmatrix} -2 & 4 & 3 \\ -3 & 0 & 3 \\ 1 & 5 & -1 \end{bmatrix} \right|$

$= \left| 3\det\begin{bmatrix} 4 & 3 \\ 5 & -1 \end{bmatrix} + 0 - 3\det\begin{bmatrix} -2 & 4 \\ 1 & 5 \end{bmatrix} \right|$

$= |3(-19) - 3(-14)| = |-15|$

$= 15$ units^3

)



THE PLANE HAS NORMAL
$\vec{n} = (-2, 1, 5)$
WE CAN USE THIS AS THE DIRECTION VECTOR FOR THE LINE.

$\therefore$ THE EQUATION OF THE LINE IS

$$(x, y, z) = (3 - 2t, -1 + t, 7 + 5t) \quad t \in \mathbb{R}$$

) $\vec{AB} = (-2, -1, 7)$, $\vec{AC} = (-1, -2, 1)$

$= \vec{AB} \times \vec{AC} = \left(\det\begin{bmatrix} -1 & -2 \\ 7 & 1 \end{bmatrix}, -\det\begin{bmatrix} -2 & -1 \\ 7 & 1 \end{bmatrix}, \det\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \right)$

$= (13, -5, 3)$

EQUATION OF THE PLANE:

$13x - 5y + 3z + d = 0$ (USING C)

$13(0) - 5(1) + 3(-3) + d = 0$

$-14 + d = 0$

$d = 14$

$\therefore \boxed{13x - 5y + 3z + 14 = 0}$

8) $\begin{bmatrix} 2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 2 & -2 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -4 & 3 & 5 \end{bmatrix}$

$\xrightarrow{R_2 \cdot (-\frac{1}{4})} \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -3/4 & -5/4 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 & -1/4 & -3/4 \\ 0 & 1 & -3/4 & -5/4 \end{bmatrix}$

LET $z = t$

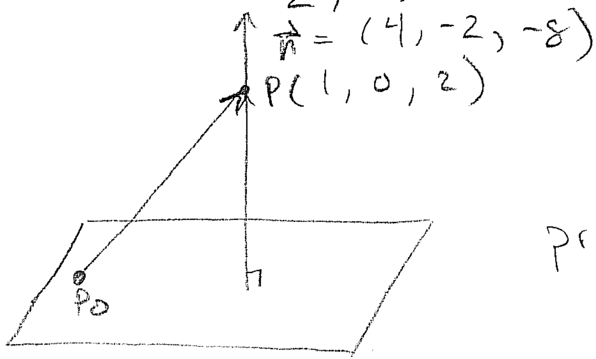
- $y - \frac{3}{4}t = \frac{-5}{4} \Rightarrow y = \frac{-5}{4} + \frac{3}{4}t$
- $x - \frac{1}{4}t = \frac{-3}{4} \Rightarrow x = \frac{-3}{4} + \frac{1}{4}t$

$\therefore (x, y, z) = \left(-\frac{3}{4} + \frac{1}{4}t, \frac{-5}{4} + \frac{3}{4}t, t\right) \quad t \in \mathbb{R}$

b) $(x, y, z) = \left(1 + \frac{1}{4}t, -4 + \frac{3}{4}t, 6 + t\right) \quad t \in \mathbb{R}$

9) $4x - 2y - 8z + 9 = 10 \Rightarrow 4x - 2y - 8z - 1 = 0$

$P_0\left(0, \frac{-1}{2}, 0\right)$ IS ON THE PLANE



$\therefore \overrightarrow{P_0P} = \left(1, \frac{1}{2}, 2\right)$

$\text{proj}_{\vec{n}} \overrightarrow{P_0P} = \frac{(1, \frac{1}{2}, 2) \cdot (4, -2, -8)}{(4, -2, -8) \cdot (4, -2, -8)} (4, -2, -8)$

$= \frac{-13}{84}(4, -2, -8)$

$\therefore$ DISTANCE $= \|\text{proj}_{\vec{n}} \overrightarrow{P_0P}\| = \left\| \frac{-13}{84}(4, -2, -8) \right\|$

$= \left| \frac{-13}{84} \right| \|(4, -2, -8)\| = \frac{13}{84}\sqrt{(4)^2 + (-2)^2 + (-8)^2}$

$= \frac{13}{84}\sqrt{84}$ units

BONUS: LET $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = -\det \begin{bmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

$$= -\vec{v} \cdot (\vec{u} \times \vec{w}) = -(\vec{u} \times \vec{w}) \cdot \vec{v}$$