Test 3

Question 1. *(5 marks)* Given $\vec{\mathbf{u}} = (-2,3,1)$, $\vec{\mathbf{v}} = (1,3,-2)$ and $\vec{\mathbf{w}} = (-3,0,4)$ find the following (if possible):

(a) $(\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})\vec{\mathbf{v}}$ **(b)** $\vec{\mathbf{v}} \times (\vec{\mathbf{u}} \cdot \vec{\mathbf{w}})$ **(c)** $(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) - \vec{\mathbf{w}}$

Question 2. *(4 marks)*
(a) The projection of $\vec{\mathbf{u}} = (5,2,-5)$ onto $\vec{\mathbf{a}} = (3,-1,2)$.
(b) Write $\vec{\mathbf{u}}$ as the sum of 2 vectors, one parallel to $\vec{\mathbf{a}}$ and one perpendicular to $\vec{\mathbf{a}}$ (indicate which is which).

Question 3. *(8 marks)* The points $A(1,-2,1)$, $B(4,-1,2)$ and $C(3,-1,6)$ are three vertices of a triangle.
(a) Let θ be the angle of the triangle at vertex B. Find $\cos\theta$.
(b) Find the area of the triangle.

Question 4. *(4 marks)* Determine whether or not the planes $2x-3y-2z+4=0$ and $x+4y-5z-7=0$ are perpendicular. Explain your answer.

Question 5. *(4 marks)* Find the equation of the line through the point $P(3,-1,7)$ and perpendicular to the plane $-2x+y+5z-13=0$.

Question 6. *(6 marks)* Given $\vec{\mathbf{u}} = (1,5,-1)$, $\vec{\mathbf{v}} = (-1,0,2)$ and $\vec{\mathbf{w}} = (-2,5,3)$
(a) Find $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}})$.
(b) Find the volume of the parallelepiped determined by $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$, and $\vec{\mathbf{w}}$.

Question 7. *(5 marks)* Find the equation of the plane passing through the points $A(1,2,0)$, $B(-1,3,2)$ and $C(0,2,1)$.

Question 8. *(6 marks)*
(a) Find the equation of the line of intersection of the planes $3x+4y-7z-4=0$ and $x+2y-2z=-2$.
(b) Find the equation of the line that is parallel to both the above planes and passing through the point $P(2,1,4)$.

Question 9. *(5 marks)* Find the distance between the plane $3x-12y+5z+9=0$ and the point $(-1,1,2)$.

Bonus. *(3 marks)* Prove that $\vec{\mathbf{u}} \cdot (\vec{\mathbf{v}} \times \vec{\mathbf{w}}) = -(\vec{\mathbf{u}} \times \vec{\mathbf{w}}) \cdot \vec{\mathbf{v}}$

TEST 3 SOLUTIONS

1) a) $(\vec{u}\cdot\vec{w})\vec{v} = \big((-2,3,1)\cdot(-3,0,4)\big)(1,3,-2) = 10(1,3,-2)$

$= (10, 30, -20)$

b) NOT POSSIBLE

c) $\vec{u}\times\vec{v} = \left(\det\begin{bmatrix}3 & 3\\ 1 & -2\end{bmatrix}, -\det\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}, \det\begin{bmatrix}-2 & 1\\ 3 & 3\end{bmatrix}\right) = (-9,-3,-9)$

$(\vec{u}\times\vec{v}) - \vec{w} = (-9,-3,-9) - (-3,0,4) = (-6,-3,-13)$

2) a) $\text{proj}_{\vec{a}}\vec{u} = \dfrac{\vec{u}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\vec{a} = \dfrac{(5,2,-5)\cdot(3,-1,2)}{(3,-1,2)\cdot(3,-1,2)}(3,-1,2)$

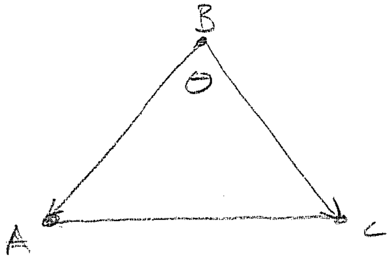
$= \dfrac{3}{14}(3,-1,2) = \left(\dfrac{9}{14}, \dfrac{-3}{14}, \dfrac{3}{7}\right)$

b) LET $\vec{w}_1 = \text{proj}_{\vec{a}}\vec{u}$

$\vec{w}_2 = \vec{u} - \vec{w}_1 = (5,2,-5) - \left(\dfrac{9}{14}, \dfrac{-3}{14}, \dfrac{3}{7}\right) = \left(\dfrac{61}{14}, \dfrac{31}{14}, \dfrac{-38}{7}\right)$

$\therefore\ \vec{u} = \underbrace{\left(\dfrac{9}{14}, \dfrac{-3}{14}, \dfrac{3}{7}\right)}_{\text{PARALLEL}} + \underbrace{\left(\dfrac{61}{14}, \dfrac{31}{14}, \dfrac{-38}{7}\right)}_{\text{PERPENDICULAR}}$

3) a)



$\vec{u} = \overrightarrow{BA} = (-3,-1,-1)$

$\vec{v} = \overrightarrow{BC} = (-1,0,4)$

$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \dfrac{-1}{\sqrt{(-3)^2+(-1)^2+(-1)^2}\,\sqrt{(-1)^2+(0)^2+(4)^2}}$

$= \dfrac{-1}{\sqrt{11}\,\sqrt{17}} = -\dfrac{1}{\sqrt{11}\sqrt{17}} = -\dfrac{1}{\sqrt{187}}$

b) $\vec{u} \times \vec{v} = \left(\det\begin{bmatrix} -1 & 0 \\ -1 & 4 \end{bmatrix}, -\det\begin{bmatrix} -3 & -1 \\ -1 & 4 \end{bmatrix}, \det\begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix} \right)$

$\begin{matrix} -3 & -1 \\ -1 & 0 \\ -1 & 4 \end{matrix} \quad = (-4, 13, -1)$

$\text{AREA} = \frac{1}{2}\|\vec{u} \times \vec{v}\| = \frac{1}{2}\sqrt{(-4)^2 + (13)^2 + (-1)^2}$

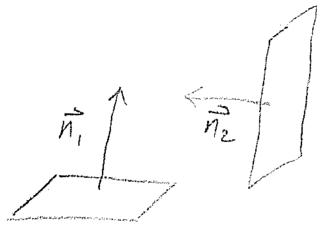
$= \frac{1}{2}\sqrt{16 + 169 + 1} = \frac{1}{2}\sqrt{186} \text{ units}^2$

4) PLANE 1 HAS NORMAL VECTOR $\vec{n}_1 = (2, -3, -2)$
PLANE 2 HAS NORMAL VECTOR $\vec{n}_2 = (1, 4, -5)$

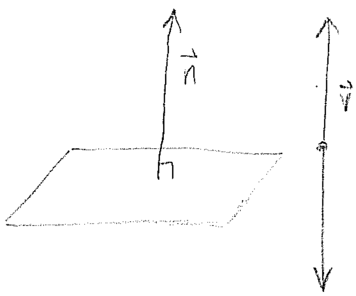
$\vec{n}_1 \cdot \vec{n}_2 = (2, -3, -2) \cdot (1, 4, -5) = 2 - 12 + 10 = 0$

$\therefore$ $\vec{n}_1$ AND $\vec{n}_2$ ARE PERPENDICULAR

$\therefore$ THE PLANES ARE PERPENDICULAR



5)



THE NORMAL VECTOR OF THE PLANE IS $\vec{n} = (-2, 1, 5)$

$\therefore$ THE LINE HAS DIRECTION VECTOR $\vec{v} = (-2, 1, 5)$

EQUATION OF THE LINE:

$(x, y, z) = (3 - 2t, -1 + t, 7 + 5t) \quad t \in \mathbb{R}$

6) a) $\vec{u}\cdot(\vec{v}\times\vec{w})$

$$\vec{v}\times\vec{w} = \left(\det\begin{bmatrix}0 & 5\\ 2 & 3\end{bmatrix}, -\det\begin{bmatrix}-1 & -2\\ 2 & 3\end{bmatrix}, \det\begin{bmatrix}-1 & -2\\ 0 & 5\end{bmatrix}\right)$$

$$= (-10, -1, -5)$$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = (1,5,-1)\cdot(-10,-1,-5) = -10-5+5$$

$$= -10$$

b) VOLUME $= |\vec{u}\cdot(\vec{v}\times\vec{w})| = |-10| = 10$ units^3

7) $\vec{AB} = (-2,1,2)$, $\vec{AC} = (1,0,-1)$

$$\vec{n} = \vec{AB}\times\vec{AC} = \left(\det\begin{bmatrix}1 & 0\\ 2 & -1\end{bmatrix}, -\det\begin{bmatrix}-2 & 1\\ 2 & -1\end{bmatrix}, \det\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}\right)$$

$$= (-1, 0, -1)$$

EQUATION OF THE PLANE:

$$ax+by+cz+d=0$$

$$-x+0y-z+d=0$$

$$-(1)+0(2)-(0)+d=0$$

$$d=1$$

$$\therefore \boxed{-x-z+1=0}$$

8) a) $3x+4y-7z=4$
$x+2y-2z=-2$

$$\Rightarrow \begin{bmatrix}3 & 4 & -7 & 4\\ 1 & 2 & -2 & -2\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix}1 & 2 & -2 & -2\\ 3 & 4 & -7 & 4\end{bmatrix} \xrightarrow{R_2-3R_1}$$

$$\begin{bmatrix}1 & 2 & -2 & -2\\ 0 & -2 & -1 & 10\end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix}1 & 0 & -3 & 8\\ 0 & -2 & -1 & 10\end{bmatrix} \xrightarrow{R_2\cdot(-\frac{1}{2})} \begin{bmatrix}1 & 0 & -3 & 8\\ 0 & 1 & \frac{1}{2} & -5\end{bmatrix}$$

LET $z=t$, $y+\frac{1}{2}t=-5$, $x-3t=8$
$y=-5-\frac{1}{2}t$, $x=8+3t$

$$\therefore \boxed{(x,y,z) = (8+3t, -5-\tfrac{1}{2}t, t) \quad t\in\mathbb{R}}$$

9) POINT ON THE PLANE $P_0(-3,0,0)$

OTHER POINT $P(-1,1,2)$

$$\overrightarrow{P_0P} = (2,1,2)$$

NORMAL VECTOR $\vec{n} = (3,-12,5)$

$$\text{proj}_{\vec{n}} \overrightarrow{P_0P} = \frac{\overrightarrow{P_0P}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} = \frac{(2,1,2)\cdot(3,-12,5)}{(3,-12,5)\cdot(3,-12,5)}(3,-12,5)$$

$$= \frac{4}{178}(3,-12,5)$$

$$\text{DISTANCE} = \|\text{proj}_{\vec{n}}\overrightarrow{P_0P}\| = \left\|\frac{4}{178}(3,-12,5)\right\|$$

$$= \left|\frac{4}{178}\right| \|(3,-12,5)\| = \frac{4}{178}\sqrt{(3)^2+(-12)^2+(5)^2}$$

$$= \frac{2}{89}\sqrt{178} = \sqrt{\frac{8}{89}}$$

BONUS: LET $\vec{u}=(u_1,u_2,u_3)$, $\vec{v}=(v_1,v_2,v_3)$, $\vec{w}=(w_1,w_2,w_3)$

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix} = -\det\begin{bmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

$$= -\vec{v}\cdot(\vec{u}\times\vec{w}) = -(\vec{u}\times\vec{w})\cdot\vec{v}$$

8) b) $\vec{v} = (3, -\frac{1}{2}, 1)$

$$(x,y,z) = (2+3t,\ 1-\tfrac{1}{2}t,\ 4+t) \quad t\in\mathbb{R}$$