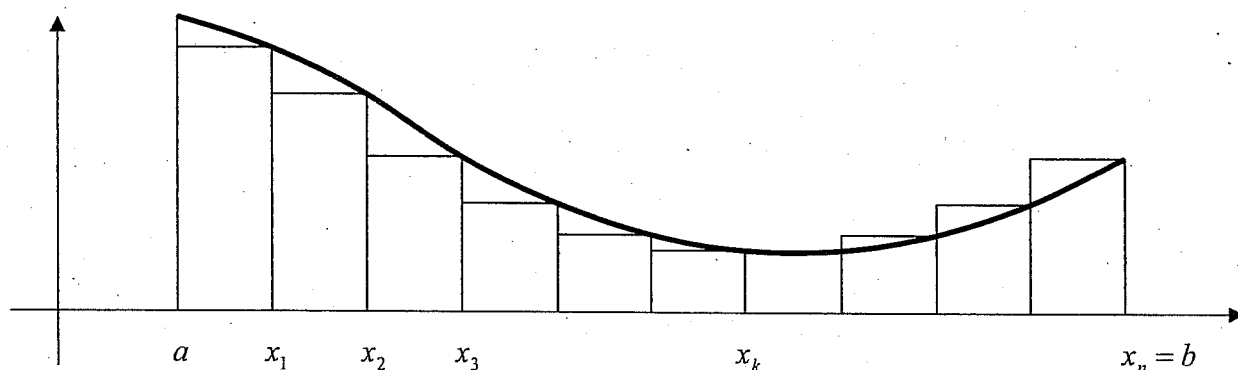# The Riemann Sum and the Definite Integral

We begin our introduction to the Riemann Sum by considering non-negative functions which are continuous over an interval $[a,b]$. To simplify the explanation and the calculations, the interval $[a,b]$ will be divided into subintervals of equal width, and the sample points will correspond to the right endpoints of the subintervals. A more general/rigorous treatment of the Riemann Sum may be found in the calculus textbook used by Pure and Applied Science students.

Let the non-negative function $y = f(x)$ be continuous over $[a,b]$. We divide $[a,b]$ into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$. The right endpoints of the subintervals are designated $x_1, x_2, x_3, \ldots, x_k, \ldots, x_n$ where $x_k = a + k\,\Delta x$ and where $x_n = b$. For each subinterval we construct a rectangle as shown in the diagram.

The base of each rectangle is Δx. The height of rectangle k (the rectangle on the subinterval with x_k as right endpoint) is $f(x_k)$. It follows that the area of rectangle k is $f(x_k)\Delta x$. The sum of the areas of all n rectangles is called the **Riemann Sum**. I.e. the Riemann Sum is equal to the expression $\displaystyle\sum_{k=1}^{n} f(x_k)\Delta x$. We see that the Riemann Sum is an approximation of the exact area under the graph of f from a to b. The larger the value of n the better the approximation. It can be proven that the limit at infinity of the Riemann Sum is the exact area under the graph of f from a to b. This limit has a special name and notation. It is called the **definite integral**.

Definition of Definite Integral If f is a continuous function defined on $[a,b]$, and if $[a,b]$ is divided into n equal subintervals of width $\Delta x = \dfrac{b-a}{n}$, and if $x_k = a + k\,\Delta x$ is the right endpoint of subinterval k, then the definite integral of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

Note: In the following two examples we consider non-negative functions on the interval $[a,b]$. As explained last page, in such cases the definite integral from a to b is the area under the curve from a to b (i.e. the area between the curve and the x-axis). The summation formulas in the appendix will be needed in the solutions of these examples.

Example 1   Use the definition of definite integral to evaluate $\displaystyle\int_0^4 \left(2x^2 + 3\right) dx$.

We subdivide the interval $[0,4]$ into n equal subintervals of width $\Delta x = \dfrac{4-0}{n} = \dfrac{4}{n}$.

Then $x_k = 0 + k \cdot \dfrac{4}{n} = \dfrac{4k}{n}$ and $f(x_k) = 2\left(\dfrac{4k}{n}\right)^2 + 3$ $\rightarrow$ $f(x_k) = \dfrac{32k^2}{n^2} + 3$.

$$f(x_k)\Delta x = \left(\dfrac{32k^2}{n^2} + 3\right)\left(\dfrac{4}{n}\right) = \dfrac{128k^2}{n^3} + \dfrac{12}{n}$$

$$\sum_{k=1}^n f(x_k)\Delta x = \sum_{k=1}^n \left(\dfrac{128k^2}{n^3} + \dfrac{12}{n}\right) = \dfrac{128}{n^3}\sum_{k=1}^n k^2 + \dfrac{12}{n}\sum_{k=1}^n 1 = \dfrac{128}{n^3}\cdot\dfrac{n(n+1)(2n+1)}{6} + \dfrac{12}{n}\cdot n$$

$$= \dfrac{128}{6}\left(\dfrac{n}{n}\right)\left(\dfrac{n+1}{n}\right)\left(\dfrac{2n+1}{n}\right) + 12 = \dfrac{64}{3}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 12$$

$$\int_0^4 \left(2x^2+3\right) dx = \lim_{n\to\infty}\sum_{k=1}^n f(x_k)\Delta x = \lim_{n\to\infty}\left[\dfrac{64}{3}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 12\right] = \dfrac{64}{3}(1)(2) + 12 = \dfrac{164}{3}$$

Example 2   Use the definition of definite integral to evaluate $\displaystyle\int_2^5 \left(8x - x^2\right) dx$.

We subdivide the interval $[2,5]$ into n equal subintervals of width $\Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}$.

Then $x_k = 2 + k \cdot \dfrac{3}{n} = 2 + \dfrac{3k}{n}$ and $f(x_k) = 8\left(2+\dfrac{3k}{n}\right) - \left(2+\dfrac{3k}{n}\right)^2 = -\dfrac{9k^2}{n^2} + \dfrac{12k}{n} + 12$.

$$f(x_k)\Delta x = \left(-\dfrac{9k^2}{n^2} + \dfrac{12k}{n} + 12\right)\left(\dfrac{3}{n}\right) = -\dfrac{27k^2}{n^3} + \dfrac{36k}{n^2} + \dfrac{36}{n}$$

$$\sum_{k=1}^n f(x_k)\Delta x = \sum_{k=1}^n \left(-\dfrac{27k^2}{n^3} + \dfrac{36k}{n^2} + \dfrac{36}{n}\right) = -\dfrac{27}{n^3}\sum_{k=1}^n k^2 + \dfrac{36}{n^2}\sum_{k=1}^n k + \dfrac{36}{n}\sum_{k=1}^n 1$$

$$= -\dfrac{27}{n^3}\cdot\dfrac{n(n+1)(2n+1)}{6} + \dfrac{36}{n^2}\cdot\dfrac{n(n+1)}{2} + \dfrac{36}{n}\cdot n = -\dfrac{9}{2}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 18\left(1+\dfrac{1}{n}\right) + 36$$

$$\int_2^5 \left(8x-x^2\right) dx = \lim_{n\to\infty}\left[-\dfrac{9}{2}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 18\left(1+\dfrac{1}{n}\right) + 36\right] = -\dfrac{9}{2}(1)(2) + 18(1) + 36 = 45$$

Until now we have only considered non-negative functions on the interval $[a,b]$. In what follows, the function may be negative on part, or all, of the interval. In those intervals where the function is negative, the value of the Riemann Sum is the <u>negative</u> of the area between the curve and the x-axis. It follows that the Riemann Sum may be a negative number.

EXERCISES Use the definition of definite integral (Riemann Sum) to evaluate each of the following definite integrals. NOTE: these functions are not necessarily non-negative. Nevertheless, the definition (Riemann Sum) may still be applied (as shown in the solutions).

1. $\displaystyle\int_{0}^{5} 4x \, dx$

2. $\displaystyle\int_{0}^{2} \left(x^{2}+10\right) dx$

3. $\displaystyle\int_{0}^{3} \left(5x^{2}-2x\right) dx$

4. $\displaystyle\int_{0}^{4} \left(-3x^{2}+5x-1\right) dx$

5. $\displaystyle\int_{4}^{6} 2x^{2} \, dx$

6. $\displaystyle\int_{1}^{5} \left(x-4x^{2}\right) dx$

7. $\displaystyle\int_{-2}^{0} \left(3x^{2}+2x\right) dx$

8. $\displaystyle\int_{-3}^{0} \left(4x^{2}-5x-1\right) dx$

9. $\displaystyle\int_{-8}^{-3} \left(4-2x^{2}\right) dx$

10. $\displaystyle\int_{-5}^{-1} \left(x^{2}+3x+5\right) dx$

11. $\displaystyle\int_{-2}^{3} \left(1-5x^{2}\right) dx$

12. $\displaystyle\int_{-1}^{2} x^{3} \, dx$

SOLUTIONS

1. $\Delta x = \dfrac{5}{n} \rightarrow x_{k} = \dfrac{5k}{n}$ and $f(x_{k}) = 4\left(\dfrac{5k}{n}\right) \rightarrow f(x_{k}) = \dfrac{20k}{n}$

$f(x_{k})\Delta x = \left(\dfrac{20k}{n}\right)\left(\dfrac{5}{n}\right) = \dfrac{100k}{n^{2}}$

$\displaystyle\sum_{k=1}^{n} f(x_{k})\Delta x = \sum_{k=1}^{n}\left(\dfrac{100k}{n^{2}}\right) = \dfrac{100}{n^{2}}\sum_{k=1}^{n} k = \dfrac{100}{2}\left(1+\dfrac{1}{n}\right)$

$\displaystyle\int_{0}^{5} 4x \, dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{k})\Delta x = \lim_{n\to\infty}\left[50\left(1+\dfrac{1}{n}\right)\right] = 50$

2. $\Delta x = \dfrac{2}{n} \rightarrow x_k = \dfrac{2k}{n}$ and $f(x_k) = \left(\dfrac{2k}{n}\right)^2 + 10 \rightarrow f(x_k) = \dfrac{4k^2}{n^2} + 10$

$$f(x_k)\Delta x = \left(\dfrac{4k^2}{n^2} + 10\right)\left(\dfrac{2}{n}\right) = \dfrac{8k^2}{n^3} + \dfrac{20}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\dfrac{8k^2}{n^3} + \dfrac{20}{n}\right) = \dfrac{8}{n^3}\sum_{k=1}^{n} k^2 + \dfrac{20}{n}\sum_{k=1}^{n} 1 = \dfrac{8}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 20$$

$$\int_0^2 (x^2 + 10)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\left[\dfrac{4}{3}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 20\right] = \dfrac{4}{3}(1)(2) + 20 = \dfrac{68}{3}$$

3. $\Delta x = \dfrac{3}{n} \rightarrow x_k = \dfrac{3k}{n}$ and $f(x_k) = 5\left(\dfrac{3k}{n}\right)^2 - 2\left(\dfrac{3k}{n}\right) \rightarrow f(x_k) = \dfrac{45k^2}{n^2} - \dfrac{6k}{n}$

$$f(x_k)\Delta x = \left(\dfrac{45k^2}{n^2} - \dfrac{6k}{n}\right)\left(\dfrac{3}{n}\right) = \dfrac{135k^2}{n^3} - \dfrac{18k}{n^2}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \dfrac{135}{n^3}\sum_{k=1}^{n} k^2 - \dfrac{18}{n^2}\sum_{k=1}^{n} k = \dfrac{135}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - \dfrac{18}{2}\left(1+\dfrac{1}{n}\right)$$

$$\int_0^3 (5x^2 - 2x)\,dx = \lim_{n\to\infty}\left[\dfrac{45}{2}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - 9\left(1+\dfrac{1}{n}\right)\right] = \dfrac{45}{2}(1)(2) - 9(1) = 36$$

4. $\Delta x = \dfrac{4}{n} \rightarrow x_k = \dfrac{4k}{n}$ and $f(x_k) = -3\left(\dfrac{4k}{n}\right)^2 + 5\left(\dfrac{4k}{n}\right) - 1 \rightarrow f(x_k) = -\dfrac{48k^2}{n^2} + \dfrac{20k}{n} - 1$

$$f(x_k)\Delta x = \left(-\dfrac{48k^2}{n^2} + \dfrac{20k}{n} - 1\right)\left(\dfrac{4}{n}\right) = -\dfrac{192k^2}{n^3} + \dfrac{80k}{n^2} - \dfrac{4}{n}$$

$$\sum_{i=1}^{n} f(x_k)\Delta x = -\dfrac{192}{n^3}\sum_{i=1}^{n} k^2 + \dfrac{80}{n^2}\sum_{i=1}^{n} k - \dfrac{4}{n}\sum_{i=1}^{n} 1 = -\dfrac{192}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + \dfrac{80}{2}\left(1+\dfrac{1}{n}\right) - 4$$

$$\int_0^4 (-3x^2 + 5x - 1)\,dx = \lim_{n\to\infty}\left[-32\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 40\left(1+\dfrac{1}{n}\right) - 4\right] = -32(1)(2) + 40(1) - 4 = -28$$

5. $\Delta x = \dfrac{2}{n} \rightarrow x_k = 4 + \dfrac{2k}{n}$ and $f(x_k) = 2\left(4 + \dfrac{2k}{n}\right)^2 \rightarrow f(x_k) = \dfrac{8k^2}{n^2} + \dfrac{32k}{n} + 32$

$$f(x_k)\Delta x = \left(\dfrac{8k^2}{n^2} + \dfrac{32k}{n} + 32\right)\left(\dfrac{2}{n}\right) = \dfrac{16k^2}{n^3} + \dfrac{64k}{n^2} + \dfrac{64}{n}$$

$$\sum_{i=1}^{n} f(x_k)\Delta x = \dfrac{16}{n^3}\sum_{i=1}^{n} k^2 + \dfrac{64}{n^2}\sum_{i=1}^{n} k + \dfrac{64}{n}\sum_{i=1}^{n} 1 = \dfrac{16}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - \dfrac{64}{2}\left(1+\dfrac{1}{n}\right) + 64$$

$$\int_4^6 2x^2\,dx = \lim_{n\to\infty}\left[\dfrac{8}{3}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 64\left(1+\dfrac{1}{n}\right)\right] = \dfrac{8}{3}(1)(2) + 32(1) + 64 = \dfrac{304}{3}$$

6. $\Delta x = \dfrac{4}{n} \rightarrow x_k = 1 + \dfrac{4k}{n}$ and $f(x_k) = \left(1 + \dfrac{4k}{n}\right) - 4\left(1 + \dfrac{4k}{n}\right)^2 \rightarrow f(x_k) = -\dfrac{64k^2}{n^2} - \dfrac{28k}{n} - 3$

$$f(x_k)\Delta x = \left(-\dfrac{64k^2}{n^2} - \dfrac{28k}{n} - 3\right)\left(\dfrac{4}{n}\right) = -\dfrac{256k^2}{n^3} - \dfrac{112k}{n^2} - \dfrac{12}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = -\dfrac{256}{n^3}\sum_{k=1}^{n} k^2 - \dfrac{112}{n^2}\sum_{k=1}^{n} k - \dfrac{12}{n}\sum_{k=1}^{n} 1 = -\dfrac{256}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - \dfrac{112}{2}\left(1+\dfrac{1}{n}\right) - 12$$

$$\int_{1}^{5}(x - 4x^2)\,dx = \lim_{n\to\infty}\left[-\dfrac{128}{3}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - 56\left(1+\dfrac{1}{n}\right) - 12\right] = -\dfrac{128}{3}(1)(2) - 56(1) - 12 = -\dfrac{460}{3}$$

7. $\Delta x = \dfrac{2}{n} \rightarrow x_k = \dfrac{2k}{n} - 2$ and $f(x_k) = 3\left(\dfrac{2k}{n} - 2\right)^2 + 2\left(\dfrac{2k}{n} - 2\right) \rightarrow f(x_k) = \dfrac{12k^2}{n^2} - \dfrac{20k}{n} + 8$

$$f(x_k)\Delta x = \left(\dfrac{12k^2}{n^2} - \dfrac{20k}{n} + 8\right)\left(\dfrac{2}{n}\right) = \dfrac{24k^2}{n^3} - \dfrac{40k}{n^2} + \dfrac{16}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \dfrac{24}{n^3}\sum_{k=1}^{n} k^2 - \dfrac{40}{n^2}\sum_{k=1}^{n} k + \dfrac{16}{n}\sum_{k=1}^{n} 1 = \dfrac{24}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - \dfrac{40}{2}\left(1+\dfrac{1}{n}\right) + 16$$

$$\int_{-2}^{0}(3x^2 + 2x)\,dx = \lim_{n\to\infty}\left[4\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - 20\left(1+\dfrac{1}{n}\right) + 16\right] = 4(1)(2) - 20(1) + 16 = 4$$

8. $\Delta x = \dfrac{3}{n} \rightarrow x_k = \dfrac{3k}{n} - 3$ and $f(x_k) = 4\left(\dfrac{3k}{n} - 3\right)^2 - 5\left(\dfrac{3k}{n} - 3\right) - 1 \rightarrow f(x_k) = \dfrac{36k^2}{n^2} - \dfrac{87k}{n} + 50$

$$f(x_k)\Delta x = \left(\dfrac{36k^2}{n^2} - \dfrac{87k}{n} + 50\right)\left(\dfrac{3}{n}\right) = \dfrac{108k^2}{n^3} - \dfrac{261k}{n^2} + \dfrac{150}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \dfrac{108}{n^3}\sum_{k=1}^{n} k^2 - \dfrac{261}{n^2}\sum_{k=1}^{n} k + \dfrac{150}{n}\sum_{k=1}^{n} 1 = \dfrac{108}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - \dfrac{261}{2}\left(1+\dfrac{1}{n}\right) + 150$$

$$\int_{-3}^{0}(4x^2 - 5x - 1)\,dx = \lim_{n\to\infty}\left[18\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) - \dfrac{261}{2}\left(1+\dfrac{1}{n}\right) + 150\right] = 18(1)(2) - \dfrac{261}{2}(1) + 150 = \dfrac{111}{2}$$

9. $\Delta x = \dfrac{5}{n} \rightarrow x_k = \dfrac{5k}{n} - 8$ and $f(x_k) = 4 - 2\left(\dfrac{5k}{n} - 8\right)^2 \rightarrow f(x_k) = -\dfrac{50k^2}{n^2} + \dfrac{160k}{n} - 124$

$$f(x_k)\Delta x = \left(-\dfrac{50k^2}{n^2} + \dfrac{160k}{n} - 124\right)\left(\dfrac{5}{n}\right) = -\dfrac{250k^2}{n^3} + \dfrac{800k}{n^2} - \dfrac{620}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = -\dfrac{250}{n^3}\sum_{k=1}^{n} k^2 + \dfrac{800}{n^2}\sum_{k=1}^{n} k - \dfrac{620}{n}\sum_{k=1}^{n} 1 = -\dfrac{250}{6}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + \dfrac{800}{2}\left(1+\dfrac{1}{n}\right) - 620$$

$$\int_{-8}^{-3}(4 - 2x^2)\,dx = \lim_{n\to\infty}\left[-\dfrac{125}{3}\left(1+\dfrac{1}{n}\right)\left(2+\dfrac{1}{n}\right) + 400\left(1+\dfrac{1}{n}\right) - 620\right] = -\dfrac{125}{3}(1)(2) + 400(1) - 620 = \dfrac{304}{3}$$

10. $\Delta x = \dfrac{4}{n} \to x_k = \dfrac{4k}{n} - 5$ and $f(x_k) = \left(\dfrac{4k}{n} - 5\right)^2 + 3\left(\dfrac{4k}{n} - 5\right) + 5 \to f(x_k) = \dfrac{16k^2}{n^2} - \dfrac{28k}{n} + 15$

$$f(x_k)\Delta x = \left(\dfrac{16k^2}{n^2} - \dfrac{28k}{n} + 15\right)\left(\dfrac{4}{n}\right) = \dfrac{64k^2}{n^3} - \dfrac{112k}{n^2} + \dfrac{60}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \dfrac{64}{n^3}\sum_{k=1}^{n} k^2 - \dfrac{112}{n^2}\sum_{k=1}^{n} k + \dfrac{60}{n}\sum_{k\to 1}^{n} 1 = \dfrac{32}{3}\left(1 + \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right) - 56\left(1 + \dfrac{1}{n}\right) + 60$$

$$\int_{-5}^{-1} (x^2 + 3x + 5)\, dx = \lim_{n\to\infty}\left[\dfrac{32}{3}\left(1 + \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right) - 56\left(1 + \dfrac{1}{n}\right) + 60\right] = \dfrac{32}{3}(1)(2) - 56(1) + 60 = \dfrac{76}{3}$$

11. $\Delta x = \dfrac{5}{n} \to x_k = \dfrac{5k}{n} - 2$ and $f(x_k) = 1 - 5\left(\dfrac{5k}{n} - 2\right)^2 \to f(x_k) = -\dfrac{125k^2}{n^2} + \dfrac{100k}{n} - 19$

$$f(x_k)\Delta x = \left(-\dfrac{125k^2}{n^2} + \dfrac{100k}{n} - 19\right)\left(\dfrac{5}{n}\right) = -\dfrac{625k^2}{n^3} + \dfrac{500k}{n^2} - \dfrac{95}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = -\dfrac{625}{n^3}\sum_{k=1}^{n} k^2 + \dfrac{500}{n^2}\sum_{k=1}^{n} k - \dfrac{95}{n}\sum_{k=1}^{n} 1 = -\dfrac{625}{6}\left(1 + \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right) + \dfrac{500}{2}\left(1 + \dfrac{1}{n}\right) - 95$$

$$\int_{-2}^{3}(1 - 5x^2)\,dx = \lim_{n\to\infty}\left[-\dfrac{625}{6}\left(1 + \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right) + 250\left(1 + \dfrac{1}{n}\right) - 95\right] = -\dfrac{625}{6}(1)(2) + 250(1) - 95 = -\dfrac{160}{3}$$

12. $\Delta x = \dfrac{3}{n} \to x_k = \dfrac{3k}{n} - 1$ and $f(x_k) = \left(\dfrac{3k}{n} - 1\right)^3 \to f(x_k) = \dfrac{27k^3}{n^3} - \dfrac{27k^2}{n^2} + \dfrac{9k}{n} - 1$

$$f(x_k)\Delta x = \left(\dfrac{27k^3}{n^3} - \dfrac{27k^2}{n^2} + \dfrac{9k}{n} - 1\right)\left(\dfrac{3}{n}\right) = \dfrac{81k^3}{n^4} - \dfrac{81k^2}{n^3} + \dfrac{27k}{n^2} - \dfrac{3}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\dfrac{81k^3}{n^4} - \dfrac{81k^2}{n^3} + \dfrac{27k}{n^2} - \dfrac{3}{n}\right) = \dfrac{81}{n^4}\sum_{k=1}^{n} k^3 - \dfrac{81}{n^3}\sum_{k=1}^{n} k^2 + \dfrac{27}{n^2}\sum_{k=1}^{n} k - \dfrac{3}{n}\sum_{k=1}^{n} 1$$

$$= \dfrac{81}{n^4}\cdot\dfrac{n^2(n+1)^2}{4} - \dfrac{81}{n^3}\cdot\dfrac{n(n+1)(2n+1)}{6} + \dfrac{27}{n^2}\cdot\dfrac{n(n+1)}{2} - \dfrac{3}{n}\cdot n$$

$$= \dfrac{81}{4}\left(1 + \dfrac{1}{n}\right)^2 - \dfrac{27}{2}\left(1 + \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right) + \dfrac{27}{2}\left(1 + \dfrac{1}{n}\right) - 3$$

$$\int_{-1}^{2} x^3\, dx = \lim_{n\to\infty}\left[\dfrac{81}{4}\left(1 + \dfrac{1}{n}\right)^2 - \dfrac{27}{2}\left(1 + \dfrac{1}{n}\right)\left(2 + \dfrac{1}{n}\right) + \dfrac{27}{2}\left(1 + \dfrac{1}{n}\right) - 3\right]$$

$$= \left[\dfrac{81}{4}(1)^2 - \dfrac{27}{2}(1)(2) + \dfrac{27}{2}(1) - 3\right] = \dfrac{15}{4}$$

APPENDIX

The following are useful formulas for working with summation notation.

1. $\displaystyle\sum_{k=1}^{n} c = nc$

2. $\displaystyle\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$

3. $\displaystyle\sum_{k=1}^{n} \left(a_k + b_k\right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

4. $\displaystyle\sum_{k=1}^{n} \left(a_k - b_k\right) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

5. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

6. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

7. $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$