CONTINUOUS PROBABILITY DISTRIBUTIONS

notes by L. Narine

Continuous Probability Distributions

A continuous random variable x can take on any value in some interval of real numbers. The distribution of this random variable is called a continuous probability distribution.

Bounded Intervals

The function $f(x)$ is a **probability density function** on the bounded interval $[a, b]$ if

1. $f(x) \geq 0$ for all x in the interval $[a, b]$, and
2. $\int_a^b f(x)\,dx = 1$

If x is a continuous random variable whose distribution is described by the function $f(x)$ on the interval $[a, b]$ then the probability that a randomly selected x lies between c and d, where $a \leq c \leq d \leq b$, is

$$P(c \leq x \leq d) = \int_c^d f(x)\,dx$$

Unbounded Intervals

The function $f(x)$ is a **probability density function** on the unbounded interval $(-\infty, \infty)$ if

1. $f(x) \geq 0$ for all x, and
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Probability Density Functions on $(-\infty, \infty)$

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

$$P(x \leq b) = P(x < b) = \int_{-\infty}^b f(x)\,dx$$

$$P(x \geq a) = P(x > a) = \int_a^{\infty} f(x)\,dx$$

Exercises

Determine whether the functions defined as follows are probability density functions on the indicated intervals. If not, explain why not.

1. $f(x) = \frac{1}{9}x - \frac{1}{18}$ on $[2,5]$

2. $f(x) = \frac{x^2}{21}$ on $[1,4]$

3. $f(x) = 4x^3$ on $[0,3]$

4. $f(x) = \frac{x^3}{81}$ on $[0,3]$

5. $f(x) = 2x^2$ on $[-1,1]$

6. $f(x) = \frac{5}{3}x^2 - \frac{5}{90}$ on $[-1,1]$

7. $f(x) = \frac{3}{13}x^2 - \frac{12}{13}x + \frac{45}{52}$ on $[0,4]$

Answers 1.Yes 2. Yes 3. No 4. N 5. No 6. No 7. No

Exercises

Find a value of k that will make the given function $f(x)$ a probability density function on the indicated interval.

1 $f(x) = k\,x^{1/2}$ on $[1,4]$

2. $f(x) = k\,x^2$ on $[-1,2]$

3. $f(x) = k\,e^{-x/2}$ on $[0,\infty)$

4. $f(x) = k\,(1+\frac{3}{\sqrt{x}})$ on $[4,9]$

Answers 1. $k = 3/14$ 2. $k = 1/3$ 3. $k = 1/2$ 4. $k = 1/11$

Exercises

Show that the functions defined below are probability density functions on the given interval; then find the indicated probabilities.

1. $f(x) = \frac{1}{2}(1+x)^{-3/2}$ on $[0,\infty)$

   a. $P(0 \le x \le 2)$ b. $P(1 \le x \le 3)$

   c. $P(x \ge 5)$

2. $f(x) = \frac{1}{2}e^{-x/2}$ on $[0,\infty)$

   a. $P(0 \le x \le 1)$ b. $P(1 \le x \le 3)$

   c. $P(x \ge 2)$

3. $f(x) = \frac{20}{(x+20)^2}$ on $[0,\infty)$

   a. $P(0 \le x \le 1)$ b. $P(1 \le x \le 5)$

   c. $P(x \ge 5)$

Answers

1 a) 0.4226 b) 0.2071 c) 0.4082 2. a) 0.3935 b) 0.3834 c) 0.3679

3 a) 0.0476 b) 0.1524 c) 0.8

Applications

1. The life x (in months) of a certain electronic computer part has a probability density function defined by

$$f(x) = \frac{1}{2}e^{-x/2} \quad \text{for all } x \text{ in } [0,\infty)$$

Find the probability that a randomly selected component will last

a. at most 12 months Ans (0.9975)

b. between 12 and 20 months (0.0024)

2. A certain type of machine has a useful life of 4 to 9 years, and its life in years has a probability density function defined by

$$f(x) = \frac{1}{11}\left(1 + \frac{3}{\sqrt{x}}\right)$$

Find the probabilities that the useful life of such a machine selected at random will last

a. longer than 6years Ans 0.5730

b. less than 5 years 0.2197

c. between 4 and 7 years 0.6250

Applications (continued)

3. The clotting time of blood is a random variable x with values from 1 second to 20 seconds and probability density function defined by

$$f(x) = \frac{1}{(\ln 20)\,x}$$

Find the following probabilities for a person selected at random:

a. The probability that the clotting time is between 1 and 5 seconds.
b. The probability that the clotting time is greater than 10 seconds.

Ans a) 0.5372 b) 0.2314

Expected value and variance of continuos random variables

If x is a continuous random variable with probability density function $f(x)$ on $[a, b]$, then the **expected value of x** is

$$E(x) = \mu = \int_a^b x\,f(x)\,dx$$

The **variance of x** is

$$Var(x) = \sigma^2 = \int_a^b (x-\mu)^2 f(x)\,dx$$

Alternate formula for the variance

$$\sigma^2 = \int_a^b x^2 f(x)\,dx - \mu^2$$

Exercises

For the given probability density functions find

a. The expected value to the nearest hundredth
b. The variance to the nearest hundredth
c. The standard deviation to the nearest hundredth
d. The probability that the random variable has a value greater than the mean
e. The probability that the value of the random variable is within one standard deviation of the mean

1. $f(x) = \frac{\sqrt{x}}{18}$ $[0, 9]$

2. $f(x) = \frac{x^{-1/3}}{6}$ $[0, 8]$

3. $f(x) = \frac{3}{2}(1 - x^2)$ $[0, 1]$

4. $f(x) = 1 - \frac{1}{\sqrt{x}}$ $[1, 4]$

5. $f(x) = \frac{3}{x^4}$ for $x \geq 1$

Answers

1.	a) 5.40	b) 5.55	c) 2.36	d) 0.54	e) 0.60
2.	a) 3.2	b) 5.76	c) 2.4	d) 0.46	e) 0.57
3.	a) 0.38	b) 0.06	c) 0.24	d) 0.46	e) 0.60
4.	a) 2.83	b) 0.57	c) 0.76	d) 0.53	e) 0.61
5.	a) 1.5	b) 0.75	c) 0.87	d) 0.30	e) 0.92

Exercises

1. The life of a certain brand of light bulb is a random variable with probability density function defined by

$$f(x) = \frac{1}{58\sqrt{x}} \quad \text{for } x \text{ in } [1,900]$$

Find
a. the expected life of such a light bulb
b. the standard deviation
c. the probability that one of these bulbs lasts longer than one standard deviation above the mean

2. The life span of a certain automobile part (in months) is a random variable with probability density function defined by

$$f(x) = \frac{1}{2}e^{-x/2} \quad \text{for } x \text{ in } [0,\infty)$$

Find
a. the expected life of this part
b. the standard deviation of the distribution
c. the probability that one of these parts lasts less than the mean number of months.

3. The age in years, of a randomly selected alcohol-impaired driver in a fatal car crash is a random variable with probability density function given by

$$f(x) = \frac{105}{4x^2} \quad \text{for } x \text{ in } [15,35]$$

Find
a. The average age of a drunk driver in a fatal crash.
b. The standard deviation of the distribution.

Answers

1.	a) 310.3 hours	b) 267 hours	c) 0.206
2.	a) 2 months	b) 2 months	c) 0.632
3.	a) 22.2years	b) 5.51 years	