Dawson College: **Calculus I**: 201-NYA-05-S06: Winter 2009

Name: Y. Lamontagne
Student ID: ____________

Test 3

This test is graded out of 41 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. Compute the indefinite integral*(1 mark each)*:

a.
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

b.
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

c.
$$\int \tan x\,dx = -\ln|\cos x| + C$$

d.
$$\int \cot x\,dx = \ln|\sin x| + C$$

e.
$$\int e^x\,dx = e^x + C$$

f.
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

g. *(bonus)*
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

h. *(bonus)*
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

Question 2. Compute the indefinite integral.

a. *(3 marks)*

$$\int \frac{z(z-1)}{\sqrt{z}}\,dz = \int \frac{z^2 - z}{\sqrt{z}}\,dz = \int \frac{z^2}{\sqrt{z}} - \frac{z}{\sqrt{z}}\,dz$$

$$= \int z^{3/2} - z^{1/2}\,dz$$

$$= \frac{z^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$

$$= \frac{2z^{5/2}}{5} - \frac{2z^{3/2}}{3} + C$$

b. *(5 marks)*

$$\int \sin \pi x \cos^7 \pi x\,dx$$

c. *(4 marks)*

$$\int \theta e^{\theta^2}\,d\theta$$

d. *(2 bonus marks)*

$$\int \frac{1}{e^x + e^{-x}}\,dx$$

b) $\displaystyle\int \sin \pi x\,(\cos \pi x)^7\,dx \overset{1}{=} \int \sin \pi x\; u^7\,dx$

$\displaystyle\overset{2}{=} \int \cancel{\sin \pi x}\; u^7 \frac{du}{-\cancel{\sin \pi x}\,\pi} = \frac{-1}{\pi}\int u^7\,du$

$\displaystyle= \frac{-1}{\pi}\frac{u^8}{8} + C$

$\displaystyle\overset{1}{=} \frac{-(\cos \pi x)^8}{8\pi} + C$

let $u \overset{1}{=} \cos \pi x$

$\displaystyle\frac{du}{dx} = \frac{d}{dx}\left[\cos \pi x\right]$

$\displaystyle\frac{du}{dx} = -\sin \pi x\;\pi$

$\displaystyle\frac{du}{-\sin \pi x\,\pi} \overset{2}{=} dx$

c) $\displaystyle\int \theta e^{\theta^2}\,d\theta$

$\displaystyle\overset{1}{=} \int \theta e^{u}\,d\theta \overset{2}{=} \int \cancel{\theta} e^{u}\frac{du}{2\theta} = \frac{1}{2}\int e^{u}\,du$

$\displaystyle= \frac{1}{2}e^{u} + C$

$\displaystyle\overset{1}{=} \frac{1}{2}e^{\theta^2} + C$

$u \overset{1}{=} \theta^2$

$\displaystyle\frac{du}{d\theta} = \frac{d}{d\theta}\left[\theta^2\right]$

$\displaystyle\frac{du}{d\theta} = 2\theta$

$\displaystyle d\theta \overset{2}{=} \frac{du}{2\theta}$

d) $\displaystyle\int \frac{1}{e^x + \frac{1}{e^x}}\,dx = \int \frac{1}{\frac{e^{2x}+1}{e^x}}\,dx = \int \frac{e^x}{1+e^{2x}}\,dx \overset{1}{=} \int \frac{e^x}{1+u^2}\,dx = \int \frac{e^x}{1+u^2}\frac{du}{e^x}$

$\displaystyle= \int \frac{1}{1+u^2}\,du$

$= \arctan u + C$

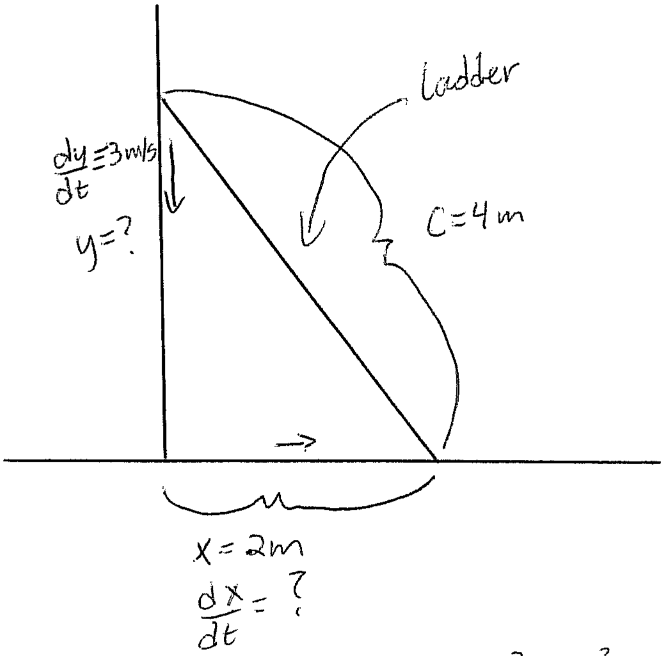
$\overset{1}{=} \arctan e^x + C$

$u \overset{1}{=} e^x$

$\displaystyle\frac{du}{dx} = e^x$

$\displaystyle\frac{du}{e^x} \overset{2}{=} dx$

Question 3. *(5 marks)* A ladder us slipping down along a vertical wall. If the ladder is 4 m long and the top of it is slipping at the constant rate of 3 m/s, how fast is the bottom of the ladder moving along the ground when the bottom is 2 m from the wall?



Use pythagorean thm to determine y

$$c^2 = x^2 + y^2$$
$$4^2 = 2^2 + y^2$$
$$16 - 4 = y^2$$
$$12 = y^2$$
$$3.46 \doteq 2\sqrt{3} = y$$

$\therefore$

$$c^2 = x^2 + y^2$$
$$\frac{d}{dt}[c^2] = \frac{d}{dt}[x^2 + y^2]$$

Since c does not depend on time

$$0 = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$
$$\frac{-y\frac{dy}{dt}}{x} = \frac{dx}{dt}$$
$$\frac{-\cancel{2}\sqrt{3} \cdot -3}{\cancel{2}} = \frac{dx}{dt}$$
$$\doteq 3\sqrt{3} = \frac{dx}{dt}$$

$\therefore$ the bottom of the ladder is moving at $3\sqrt{3}$ m/s

Question 4. Let $f(x) = x(x-4)^3$

a. *(2 marks)* Find the x and y intercepts of $f(x)$.
b. *(2 marks)* Find $f'(x)$ and solve for the critical points.
c. *(2 marks)* On what intervals is $f(x)$ increasing/decreasing?
d. *(1 mark)* Identify the relative minimum and maximum.
e. *(1 mark)* Find $f''(x)$.
f. *(2 marks)* On what intervals is $f(x)$ concave up/down?
g. *(1 mark)* Identify any inflection points.
h. *(2 marks)* Sketch the graph of $f(x)$.

a) x-intercept: $0 = f(x)$

$0 = x(x-4)^3$

$x = 0 \qquad x = 4$

y-intercept: $(0, f(0)) = (0, 0)$

b) $f'(x) = (x-4)^3 + 3x(x-4)^2$

$= (x-4)^2(x-4+3x)$

$= (x-4)^2(4x-4)$

$= 4(x-4)^2(x-1)$

Critical points:

$0 = f'(x)$

$0 = 4(x-4)^2(x-1)$

$x = 4 \qquad x = 1$

$\therefore$ critical points are $x = 1$ and $x = 4$

c)

	$(-\infty, 1)$	$(1, 4)$	$(4, \infty)$
test point, p	0	2	5
$f'(p)$	-64 $-$	16 $+$	16 $+$
inc/dec	$\searrow$	$\nearrow$	$\nearrow$

d) @ $x = 1$

$y = f(1) = 1(1-4)^3 = -27$

$\therefore (1, -27)$ a minimum

e) $f''(x) = 4 \cdot 2(x-4)(x-1) + 4(x-4)^2$

$= (x-4)[8(x-1) + 4(x-4)]$

$= (x-4)[8x - 8 + 4x - 16]$

$= (x-4)[12x - 24]$

f) $0 = f''(x)$

$0 = (x-4)[12x-24]$

$x = 4 \qquad x = 2$

	$(-\infty, 2)$	$(2, 4)$	$(4, \infty)$
test point, p	0	3	5
$f''(p)$	96 $+$	-12 $-$	36 $+$
Concavity	$\cup$	$\cap$	$\cup$

g) @ $x = 2$

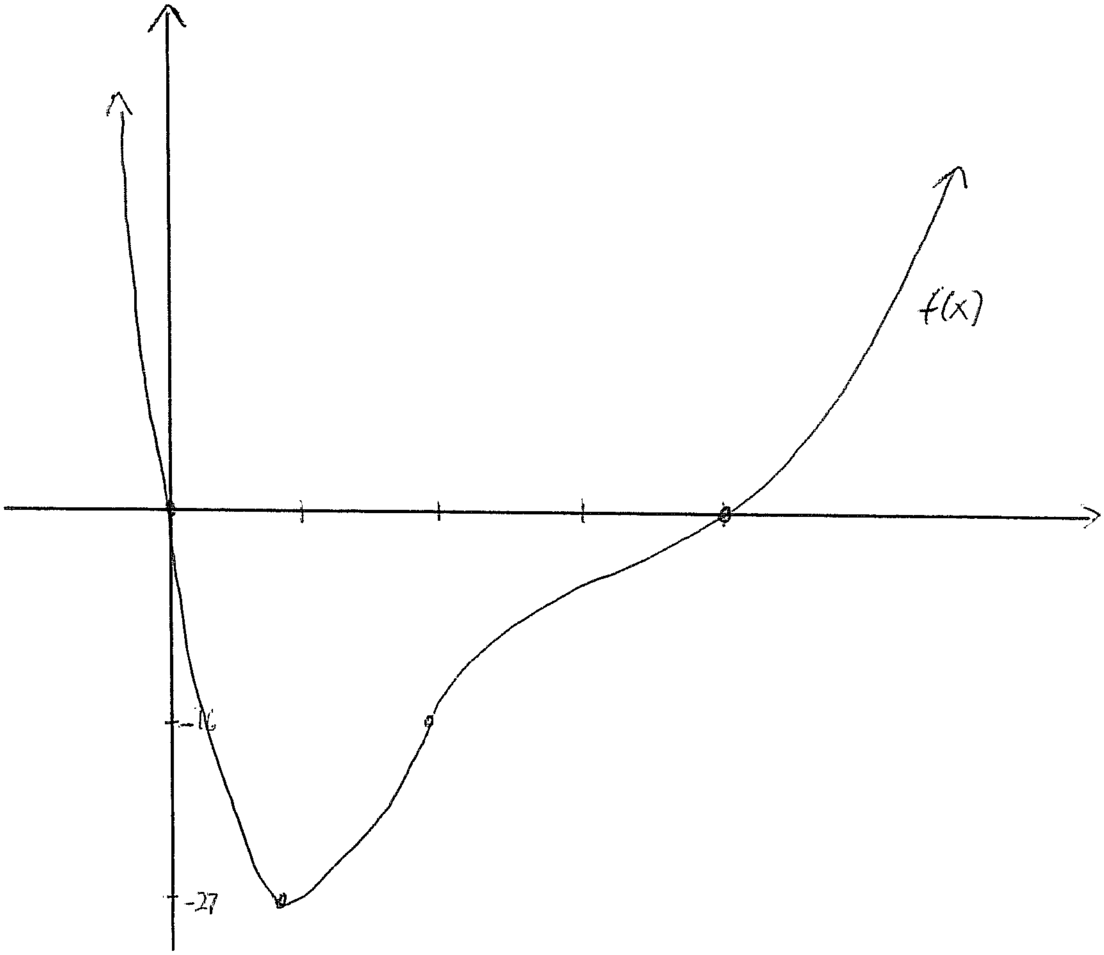
$y = f(2) = 2(2-4)^3 = -16$

@ $x = 4$

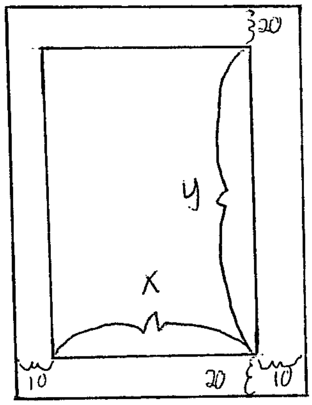
$y = f(4) = 4(4-4)^2 = 0$

$\therefore$ at $(2, -16)$ and $(4, 0)$ we have inflection points.

h)



Question 5. *(5 marks)* A rectangular building covering 7 000 m^2 is to be built on a rectangular lot. If the building is to be 10 m from the boundary lot on each side and 20 m from the boundary in front and back, find the dimensions of the building if the area of the lot is a minimum.



① $xy = 7000$

② $A = (20+20+y)(10+10+x)$

$= (40+y)(20+x)$

$= 800 + 40x + 20y + xy$

From ① isolate y $y = \frac{7000}{x}$ sub into ②

$$A = 800 + 40x + 20\left(\frac{7000}{x}\right) + \cancel{x}\left(\frac{7000}{\cancel{x}}\right)$$

$$A(x) = 7800 + 40x + \frac{140000}{x}$$

$$A'(x) = 40 - \frac{140000}{x^2}$$

Lets find the critical points

$0 = A'(x)$

$0 = 40 - \frac{140000}{x^2}$

$\frac{140000}{x^2} = 40$

$\frac{140000}{40} = x^2$

$x^2 = 3500$

$x = \pm\sqrt{3500}$

$x = \sqrt{3500}$ m only possibility

$= 10\sqrt{35}$

Lets verify that it is a minimum.

$$A''(x) = \frac{+28000}{x^3}$$

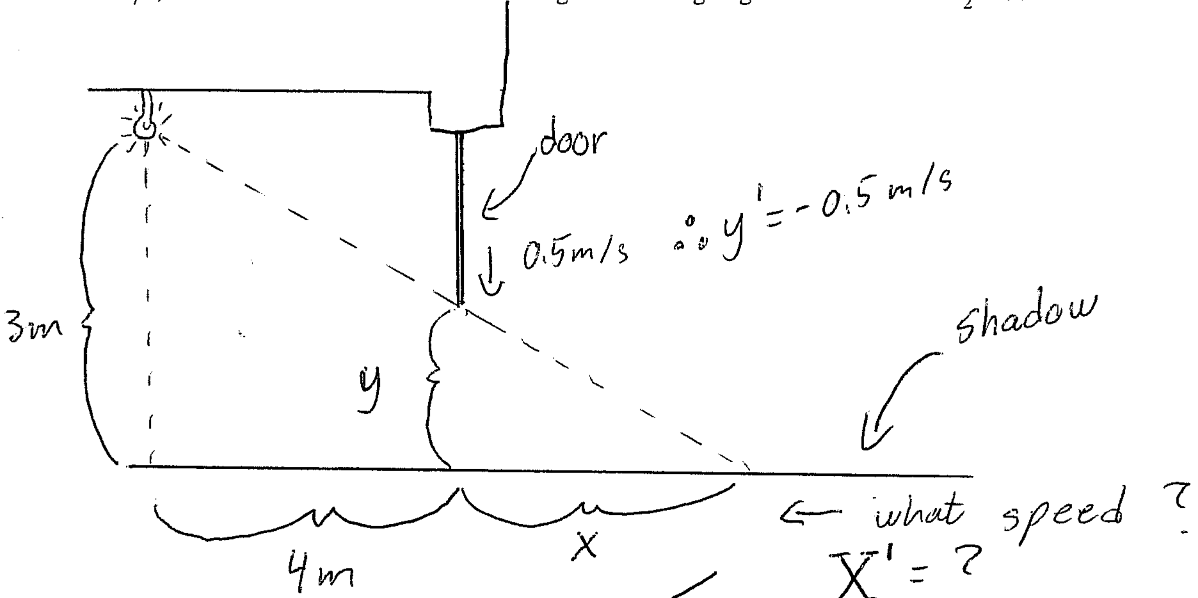
$$A''(\sqrt{350}) = \frac{28000}{(\sqrt{350})^3} > 0$$

$\therefore$ a minimum

$\therefore x = 10\sqrt{35}$ m

and $y = 20\sqrt{35}$ m

Bonus. *(3 marks)* A light in a garage is 3 m above the floor and 4 m behind the door. If the garage door descends vertically at 0.5 m/s, how fast is the door's shadow moving toward the garage when the door is $\frac{\sqrt{3}}{2}$ above the floor.



$X' = ?$

$X = ?$

$X = 4 + x$

$$\frac{Y}{X} = \frac{y}{x}$$

$$\frac{3}{X} = \frac{y}{X-4}$$

from

$$3(X-4) = yX$$

$$3X - 12 = yX$$

$$\frac{d}{dt}[3X-12] = \frac{d}{dt}[yX]$$

$$3X' = yX' + y'X$$

$$X'(3-y) = y'X$$

$$X' = \frac{y'X}{3-y}$$

Lets find X

$$\frac{Y}{X} = \frac{y}{X-4}$$

$$\frac{3}{X} = \frac{\frac{\sqrt{3}}{2}}{X-4}$$

$$3X - 12 = X\frac{\sqrt{3}}{2}$$

$$X\left(3 - \frac{\sqrt{3}}{2}\right) = 12$$

$$X = \frac{12}{\frac{6-\sqrt{3}}{2}} = \frac{24}{6-\sqrt{3}}$$

$$X' = \frac{-\frac{1}{2}\left(\frac{24}{6-\sqrt{3}}\right)}{3 - \frac{\sqrt{3}}{2}}$$

12

$$= \frac{-\frac{12}{6-\sqrt{3}}}{\frac{6-\sqrt{3}}{2}} = \frac{-24}{(6-\sqrt{3})^2} \text{ m/s}$$