Dawson College: **Calculus I**: 201-NYA-05-S06: Winter 2009

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Test 4

This test is graded out of 32 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work.

Question 1. Compute the definite integral:

a. *(4 marks)*

$$\int_0^{1/\sqrt{2}} \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx$$

b. *(4 marks)*

$$\int_0^1 \frac{x^2+x^3}{4x^3+3x^4+303}\,dx$$

$u \overset{①}{=} \arcsin x \qquad u(0) = 0$

$u(1/\sqrt{2}) = \arcsin\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$dx \overset{②}{=} \sqrt{1-x^2}\,du$

$\overset{①}{=} \int_0^{1/\sqrt{2}} \frac{e^u}{\sqrt{1-x^2}}\,dx$

$\overset{②}{=} \int_0^{\pi/4} \frac{e^u}{\cancel{\sqrt{1-x^2}}} \cancel{\sqrt{1-x^2}}\,du$

$= \int_0^{\pi/4} e^u\,du$

$= \left[e^u\right]_0^{\pi/4} = e^{\pi/4} - e^0$

$= e^{\pi/4} - 1$

$u \overset{①}{=} 4x^3 + 3x^4 + 303 \qquad u(0) = 303$

$\frac{du}{dx} = 12x^2 + 12x^3 \qquad u(1) = 310$

$dx \overset{②}{=} \frac{du}{12(x^2+x^3)}$

$\overset{①}{=} \int_0^1 \frac{x^2+x^3}{u}\,dx$

$\overset{②}{=} \int_{303}^{310} \frac{\cancel{x^2+x^3}}{u} \frac{du}{12(\cancel{x^2+x^3})}$

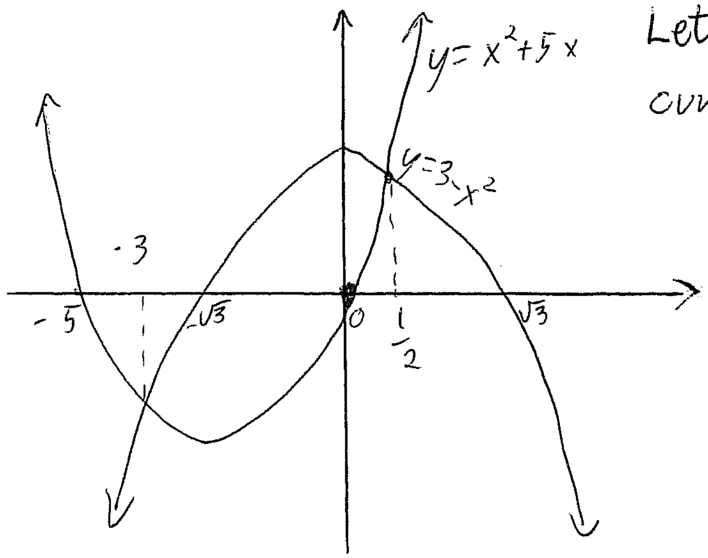
$= \frac{1}{12}\int_{303}^{310} \frac{1}{u}\,du$

$= \frac{1}{12}\left[\ln|u|\right]_{303}^{310} = \frac{1}{12}\left[\ln 310 - \ln 303\right]$

$= \frac{1}{12} \ln \frac{310}{303}$

$= \ln \sqrt[12]{\frac{310}{303}}$

Question 2. *(5 marks)* Find the area of the region bounded by the graphs of $y = x^2 + 5x$ and $y = 3 - x^2$.



Lets find where the two curves intersect.

$$x^2+5x = 3-x^2$$

$$2x^2+5x-3=0$$

$$(2x-1)(x+3)=0$$

$2x-1=0$ $\quad$ $x=-3$

$x=\frac{1}{2}$

$$\text{Area} = \int_{-3}^{\frac{1}{2}} \text{Top curve} - \text{bottom curve}\, dx$$

$$= \int_{-3}^{\frac{1}{2}} 3-x^2 - [x^2+5x]\, dx$$

$$= \int_{-3}^{\frac{1}{2}} 3-5x-2x^2\, dx$$

$$= \left[3x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_{-3}^{\frac{1}{2}}$$

$$= 3\left(\tfrac{1}{2}\right) - \frac{5}{2}\left(\tfrac{1}{2}\right)^2 - \frac{2}{3}\left(\tfrac{1}{2}\right)^3 - 3(-3) + \frac{5}{2}(-3)^2 + \frac{2}{3}(-3)^3$$

$$= \frac{3}{2} - \frac{5}{8} - \frac{2}{24} + 9 + \frac{45}{2} - \frac{54}{3}$$

$$= \frac{343}{24}$$

Question 3.

a. *(5 marks)* Use the Trapezoidal Rule with $n = 4$ to approximate the area under the curve

$$f(x) = x\sqrt{x^2+1}$$

on $[0,1]$.

b. *(4 marks)* Find the exact solution.

a) $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

$x_i = a + i\Delta x$

$x_0 = 0 + 0\Delta x = 0$

$x_1 = 0 + 1\Delta x = \frac{1}{4}$

$x_2 = \ldots = \frac{2}{4}$

$x_3 = \ldots = \frac{3}{4}$

$x_4 = \quad = 1$

$$\int_0^1 x\sqrt{x^2+1}\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$$= \frac{1}{8}\left[0 + 2\left(\tfrac{1}{4}\right)\sqrt{\left(\tfrac{1}{4}\right)^2+1} + 2\left(\tfrac{3}{4}\right)\sqrt{\left(\tfrac{2}{4}\right)^2+1} + 2\left(\tfrac{3}{4}\right)\sqrt{\left(\tfrac{3}{4}\right)^2+1} + 1\sqrt{1^2+1}\right]$$

$$= \frac{1}{8}\left[\frac{1}{2}\sqrt{\frac{17}{16}} + \sqrt{\frac{5}{4}} + \frac{3}{2}\sqrt{\frac{25}{16}} + \sqrt{2}\right]$$

$$= \frac{1}{8}\left[\frac{\sqrt{17}}{8} + \frac{\sqrt{5}}{2} + \frac{15}{8} + \sqrt{2}\right]$$

$\doteq 0.615$

b) $\int_0^1 x\sqrt{x^2+1}\,dx$

$u \overset{①}{=} x^2 + 1 \qquad u(0) = 0^2 + 1 = 1$

$\frac{du}{dx} = 2x \qquad u(1) = 1^2 + 1 = 2$

$\frac{du}{2x} \overset{②}{=} dx$

$$\overset{①,②}{=} \int_1^2 x\sqrt{u}\,\frac{du}{2x} = \frac{1}{2}\int_1^2 \sqrt{u}\,du$$

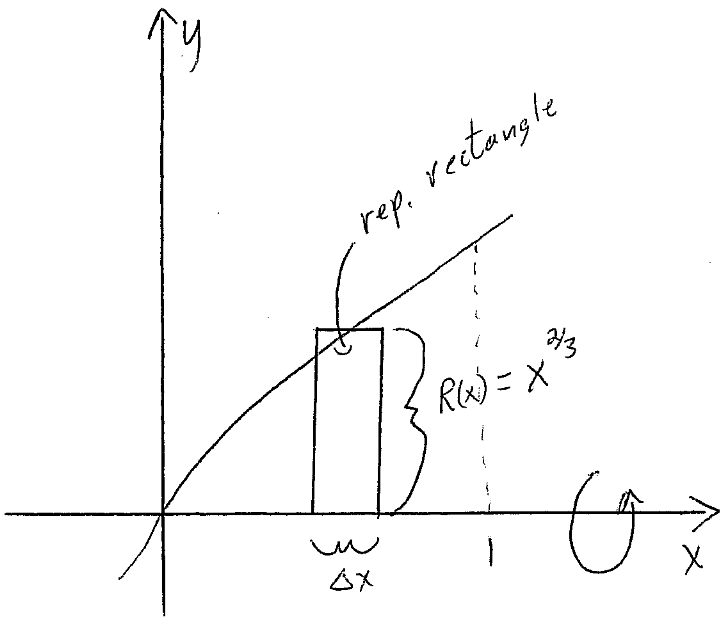
$$= \frac{1}{2}\left[\frac{2u^{3/2}}{3}\right]_1^2$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)\left[2^{3/2} - 1^{3/2}\right]$$

$$= \frac{1}{2}\left(\frac{2}{3}\right)\left(\sqrt{8} - 1\right)$$

$$= \frac{1}{3}\left(\sqrt{8} - 1\right) \doteq 0.609$$

Question 4. *(5 marks)* Using the disk method, find the volume of the solid obtained by rotating the region bounded by the graphs of the functions $y = x^{2/3}$, $y = 0$, $x = 0$ and $x = 1$ about the x-axis.



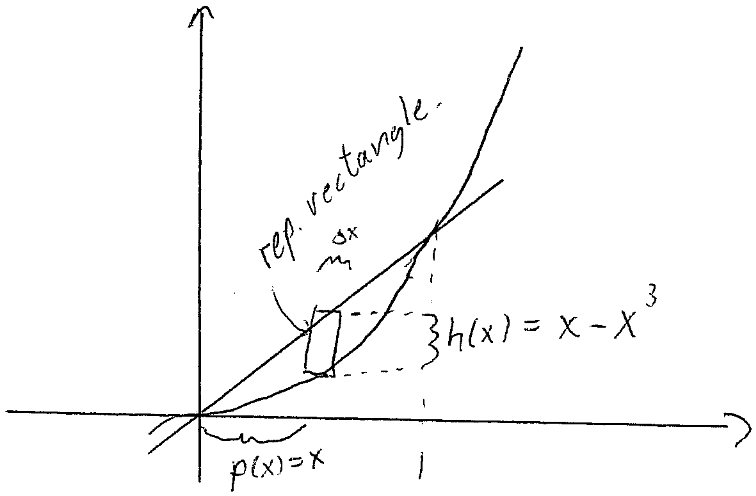
rep. element:

$$\Delta V = \pi \left(R(x)\right)^2 \Delta x$$
$$= \pi \left(x^{2/3}\right)^2 \Delta x$$
$$= \pi x^{4/3} \Delta x$$

$$\text{Volume} = \int_0^1 \pi x^{4/3}\, dx$$
$$= \pi \left[\frac{3x^{7/3}}{7}\right]_0^1$$
$$= \pi \frac{3}{7}(1)^{7/3}$$
$$= \frac{3\pi}{7}$$

$\therefore$ volume is $\frac{3\pi}{7}$

Question 5. *(5 marks)* Using the shell method, find the volume of the solid obtained by rotating the region bounded by the graphs of the functions $y = x^3$, $y = x$, $x = 0$ and $x = 1$ about the y-axis.



rep. element:

$$\Delta V = 2\pi p(x) h(x) \Delta x$$
$$= 2\pi x (x - x^3) \Delta x$$

$$\text{Volume} = \int_0^1 2\pi x (x - x^3)\,dx$$

$$= \int_0^1 2\pi (x^2 - x^4)\,dx$$

$$= 2\pi \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1$$

$$= 2\pi \left[\frac{1}{3} - \frac{1}{5} \right] = 2\pi \left[\frac{5}{15} - \frac{3}{15} \right]$$

$$= 2\pi \left[\frac{2}{15} \right]$$

$$= \frac{4\pi}{15}$$

Bonus. *(1 mark each)* Integrate the following indefinite integrals.

a.
$$\int \frac{1}{x^2+4x+5}\,dx$$

complete the square

$$\int \frac{1}{(x+2)^2+1}\,dx$$

$u = x+2$

$du = dx$

$$= \int \frac{1}{u^2+1}\,du = \arctan u + C$$

$$= \arctan(x+2) + C$$

b.
$$\int \frac{\sqrt{x}-1}{\sqrt{x}+1}\,dx$$

$u \overset{①}{=} \sqrt{x}+1$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$2\sqrt{x}\,du \overset{②}{=} dx$

$\sqrt{x} = u-1$ from ①

$$\overset{①,②}{=} \int \frac{(\sqrt{x}-1)\cdot 2\sqrt{x}\,du}{u}$$

$$= \int \frac{(u-1-1)\,2(u-1)\,du}{u}$$

$$= 2\int \frac{(u-2)(u-1)}{u}\,du$$

$$= 2\int \frac{u^2-3u+2}{u}\,du$$

$$= 2\int u - 3 + \frac{2}{u}\cdot du$$

$$= 2\left[\frac{u^2}{2} - 3u + 2\ln|u|\right] + C$$

$$= (\sqrt{x}+1)^2 - 6(\sqrt{x}+1) + 4\ln|\sqrt{x}+1| + C$$

c.
$$\int \frac{\cos^5 x}{\sqrt{\sin x}}\,dx$$

note:

$\cos^2 x = 1 - \sin^2 x$

$$= \int \sin^{-\frac{1}{2}} x \cos x \cos^4 x\,dx$$

$$= \int \sin^{-\frac{1}{2}} x \cos x\,(1-\sin^2 x)^2\,dx$$

$$= \int \sin^{-\frac{1}{2}} x \cos x\left[1 - 2\sin^2 x + \sin^4 x\right]dx$$

$$= \int \cos x\left[\sin^{-\frac{1}{2}} x - 2\sin^{\frac{3}{2}} x + \sin^{\frac{7}{2}} x\right]dx$$

$u \overset{①}{=} \sin x$

$\frac{du}{dx} = \cos x$

$\frac{du}{\cos x} \overset{②}{=} dx$

$$\overset{①,②}{=} \int u^{-\frac{1}{2}} - 2u^{3/2} + u^{7/2}\,du$$

$$= 2u^{\frac{1}{2}} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C$$

$$= 2\sqrt{\sin x} - \frac{4}{5}\sin^{5/2} x + \frac{2}{9}\sin^{9/2} x + C$$