WINTER 2005 FINAL EXAM

1) i) $\int x\sqrt{x-1}\,dx$

LET $u = x-1$
$du = dx$
$\Rightarrow x = u+1$

$= \int (u+1)\sqrt{u}\,du$

$= \int u^{3/2} + u^{1/2}\,du = \dfrac{u^{5/2}}{5/2} + \dfrac{u^{3/2}}{3/2} + C$

$= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C$

ii) $x^2+4x+5 = (x^2+4x+4)+5-4 = (x+2)^2+1$

$\therefore \int \dfrac{1}{x^2+4x+5}\,dx = \int \dfrac{1}{(x+2)^2+1}\,dx$

LET $u = x+2$
$du = dx$

$= \int \dfrac{1}{u^2+1}\,du = \arctan u + C$

$= \arctan(x+2) + C$

iii) $\int \sqrt{x}\ln x\,dx = uv - \int v\,du$

LET $u = \ln x$ $\quad dv = x^{1/2}dx$
$du = \frac{1}{x}dx$ $\quad v = \frac{2}{3}x^{3/2}$

$= \frac{2}{3}x^{3/2}\ln x - \int \frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx$

$= \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\int x^{1/2}\,dx = \frac{2}{3}x^{3/2}\ln x - \frac{2}{3}\dfrac{x^{3/2}}{3/2} + C$

$= \frac{2}{3}x^{3/2}\ln x - \frac{4}{9}x^{3/2} + C$

iv) $\int \sin^2 x \cos^3 x \, dx = \int \sin^2 x \cos^2 x \cos x \, dx$

LET $u = \sin x$
$du = \cos x \, dx$

$= \int \sin^2 x (1 - \sin^2 x) \cos x \, dx$

$= \int u^2 (1-u^2) \, du = \int u^2 - u^4 \, du$

$= \frac{u^3}{3} - \frac{u^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

v) $\frac{5x^2+12}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$

$\Rightarrow 5x^2+12 = A(x^2+4) + (Bx+C)x = Ax^2 + 4A + Bx^2 + Cx$

$= (A+B)x^2 + Cx + 4A$

$\Rightarrow A+B = 5, \quad C = 0, \quad 4A = 12$

$\Rightarrow A = 3, \quad B = 2, \quad C = 0$

$\therefore \int \frac{5x^2+12}{x(x^2+4)} \, dx = \int \frac{3}{x} + \frac{2x}{x^2+4} \, dx$

$= 3\ln|x| + \ln(x^2+4) + c$

vi) $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}\,dx$

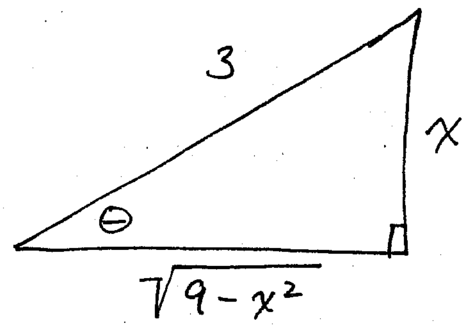
LET $x = 3\sin\theta$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\sqrt{\cos^2\theta}$

$= 3|\cos\theta| = 3\cos\theta$ (since $\cos\theta \ge 0$ on $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$)

$dx = 3\cos\theta\,d\theta$

$\sin\theta = \frac{x}{3}$



$$= \int \frac{9\sin^2\theta}{3\cos\theta}\,3\cos\theta\,d\theta$$

$$= 9\int \sin^2\theta\,d\theta =$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}\left(\theta - \frac{2\sin\theta\cos\theta}{2}\right) + C$$

$$= \frac{9}{2}\left(\arcsin\left(\frac{x}{3}\right) - \left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)\right) + C$$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C$$

2) • $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ • $x_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$

• $f(x_i) = (x_i)^2 - 3 = \left(\frac{2i}{n}\right)^2 - 3 = \frac{4i^2}{n^2} - 3$

• $f(x_i)\Delta x = \left(\frac{4i^2}{n^2} - 3\right)\left(\frac{2}{n}\right) = \frac{i^2}{n^3} - \frac{6}{n}$

• $\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{8i^2}{n^3} - \frac{6}{n}\right) = \frac{8}{n^3}\sum_{i=1}^{n} i^2 - \frac{6}{n}\sum_{i=1}^{n} 1$

$= \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} - \frac{6}{n} \cdot n$

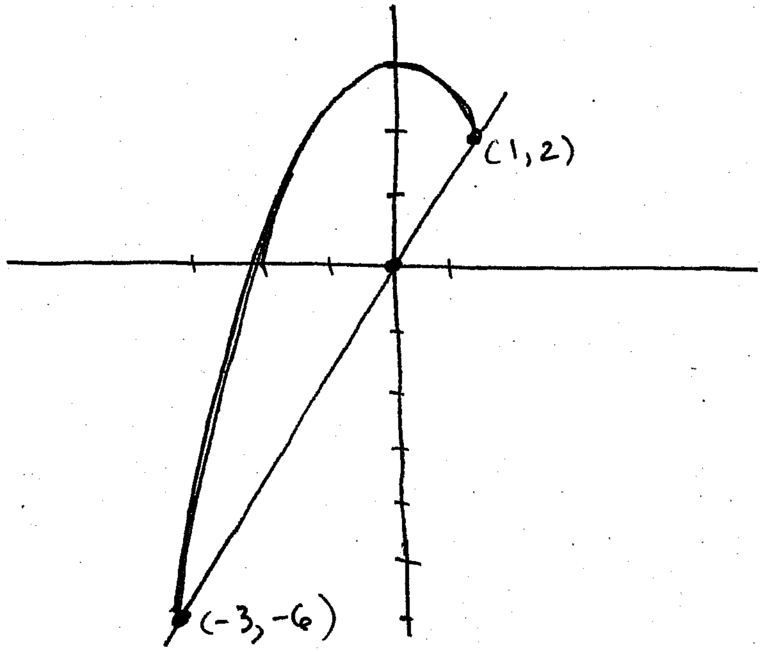
$= \frac{4}{3}\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right) - 6$

$\int_0^2 (x^2-3)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 6\right]$

$= \frac{4}{3}\cdot 1 \cdot 2 - 6 = \frac{8}{3} - 6 = \frac{-10}{3}$

3) $-x^2+3 = 2x \Rightarrow 0 = x^2+2x-3 = (x+3)(x-1)$

$\Rightarrow x = 1, -3$



AREA $= \int_{-3}^{1} f(x) - g(x)\,dx = \int_{-3}^{1} (-x^2+3) - 2x\,dx$

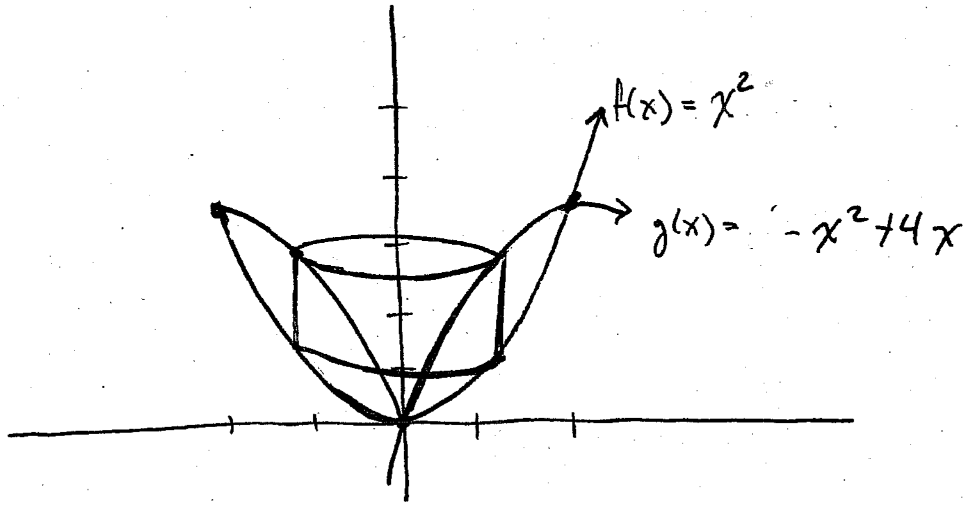
$= \left[-\frac{x^3}{3} + 3x - x^2 \right]_{-3}^{1}$

$= \left(-\frac{1}{3} + 3 - 1\right) - \left(-\frac{(-3)^3}{3} + 3(-3) - (-3)^2\right)$

$= -\frac{1}{3} + 2 - 9 + 9 + 9 = \frac{32}{3}$ units^2

4) $-x^2+4x = x^2 \Rightarrow 2x^2-4x=0 \Rightarrow 2x(x-2)=0$

$\therefore\ x = 0, 2$



$A(x) = 2\pi r h$, $r = x$, $h = (-x^2+4x) - x^2$

$= -2x^2+4x$

$\therefore\ A(x) = 2\pi x(-2x^2+4x)$

$= \pi(-4x^3+8x^2)$

$\therefore\ V = \int_0^2 \pi(-4x^3+8x^2)\,dx$

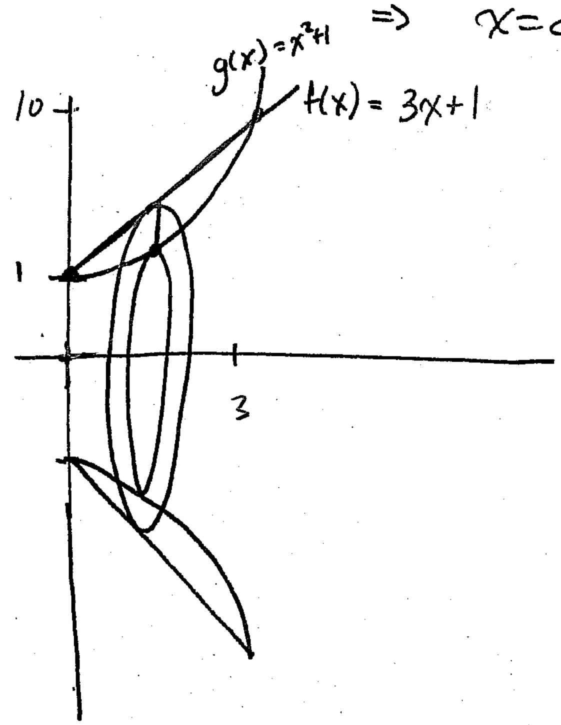
$= \pi\left[-x^4 + \frac{8}{3}x^3\right]_0^2$

$= \pi\left[-(2)^4 + \frac{8}{3}(2)^3\right] - \pi(0)$

$= \pi\left[-16 + \frac{64}{3}\right] = \pi\left(\frac{16}{3}\right)$ units^3

5) INTERSECTION! $3x+1 = x^2+1 \Rightarrow x^2-3x=0 \Rightarrow x(x-3)=0$

$\Rightarrow x=0, 3$



$A(x) = \pi R^2 - \pi r^2$

$R = 3x+1$

$r = x^2+1$

$A(x) = \pi(3x+1)^2 - \pi(x^2+1)^2$

$= \pi(9x^2+6x+1)$

$- \pi(x^4+2x^2+1)$

$= \pi(-x^4+7x^2+6x)$

$\therefore V = \int_0^3 \pi(-x^4+7x^2+6x)\,dx$

$= \pi\left[-\frac{x^5}{5} + \frac{7}{3}x^3 + 3x^2\right]_0^3$

$= \pi\left[\frac{-(3)^5}{5} + \frac{7}{3}(3)^3 + 3(3)^2\right] - \pi[0]$

$= \pi\left[\frac{-243}{5} + 63 + 27\right]$

$= \frac{207}{5}\pi$ units^3

6) i) $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \text{"}\infty\cdot 0\text{"}$

$$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \text{"}\frac{0}{0}\text{"} \overset{H}{=} \lim_{x\to\infty} \frac{\cos\left(\frac{1}{x}\right)\left(-\cancel{\frac{1}{x^2}}\right)}{\cancel{-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1$$

ii) $\lim_{x\to\infty} \left(1-\frac{2}{x}\right)^{3x} = \text{"}1^{\infty}\text{"}$

LET $y = \left(1-\frac{2}{x}\right)^{3x} \Rightarrow \ln y = 3x\ln\left(1-\frac{2}{x}\right)$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 3x\ln\left(1-\frac{2}{x}\right) = \text{"}\infty\cdot 0\text{"}$$

$$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{2}{x}\right)}{\frac{1}{3x}} = \text{"}\frac{0}{0}\text{"} \overset{H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1-\frac{2}{x}}\right)\cdot\left(\frac{2}{x^2}\right)}{-\frac{1}{3x^2}}$$

$$= \lim_{x\to\infty} \left(\frac{1}{1-\frac{2}{x}}\right)\cdot(-6) = -6$$

$$\therefore \lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln y} = e^{\lim_{x\to\infty}\ln y} = e^{-6}$$

7i) $\displaystyle\int_0^2 \frac{1}{\sqrt{2-x}}\,dx = \lim_{t\to 2^-}\int_0^t \frac{1}{\sqrt{2-x}}\,dx$

$\displaystyle = \lim_{t\to 2^-}\left[-2(2-x)^{1/2}\right]_0^t = \lim_{t\to 2^-}\left[-2(2-t)^{1/2} - (-2(2-0)^{1/2})\right]$

$= 0 + 2(2)^{1/2} = 2(2)^{1/2} = 2^{3/2}$ CONVERGES

ii) $\displaystyle\int_0^\infty e^{-3x}\,dx = \lim_{t\to\infty}\int_0^t e^{-3x}\,dx$

$\displaystyle = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_0^t = \lim_{t\to\infty}\left[-\frac{1}{3}e^{-3t} + \frac{1}{3}e^{0}\right]$

$= 0 + \frac{1}{3} = \frac{1}{3}$ CONVERGES

8) $f'(x) = \frac{x^3}{2} - \frac{1}{2x^3}$, $(f'(x))^2 = \frac{x^6}{4} - \frac{1}{2} + \frac{1}{4x^6}$

$\therefore\ 1 + (f'(x))^2 = \frac{x^6}{4} + \frac{1}{2} + \frac{1}{4x^6} = \left(\frac{x^3}{2} + \frac{1}{2x^3}\right)$

$\displaystyle\therefore\ L = \int_1^2 \sqrt{1+(f'(x))^2}\,dx = \int_1^2 \sqrt{\left(\frac{x^3}{2} + \frac{1}{2x^3}\right)^2}\,dx$

$\displaystyle = \int_1^2 \frac{x^3}{2} + \frac{1}{2}x^{-3}\,dx = \left[\frac{x^4}{8} - \frac{1}{4x^2}\right]_1^2$

$= \left[\frac{2^4}{8} - \frac{1}{4(2)^2}\right] - \left[\frac{1}{8} - \frac{1}{4}\right] = \left[2 - \frac{1}{16}\right] - \left[\frac{1}{8} - \frac{1}{4}\right]$

$= \frac{32}{16} - \frac{1}{16} - \frac{2}{16} + \frac{4}{16} = \frac{33}{16}$ units

9) $\sum_{n=1}^{\infty} \frac{2^n}{3^{n+2}} = \sum_{n=1}^{\infty} \frac{2^{n-1} \cdot 2}{3^{n-1}\ 3^3} = \sum_{n=1}^{\infty} \frac{2}{27}\left(\frac{2}{3}\right)^{n-1}$

(GEOMETRIC SERIES $|r| = \frac{2}{3} < 1$
$\therefore$ CONVERGES)

$$= \frac{2/27}{1 - 2/3} = \frac{2}{27} \cdot \frac{3}{1} = \frac{2}{9}$$

10) i) $\int_3^{\infty} \frac{1}{x(\ln x)^2}\, dx = \lim_{t\to\infty} \int_3^t \frac{1}{x(\ln x)^2}\, dx$

$u = \ln x$
$du = \frac{1}{x}\, dx$
$x = 3 \Rightarrow u = \ln 3$
$x = t \Rightarrow u = \ln t$

$$\lim_{t\to\infty} \int_{\ln 3}^{\ln t} \frac{1}{u^2}\, du$$

$$= \lim_{t\to\infty} \left[-\frac{1}{u}\right]_{\ln 3}^{\ln t} = \lim_{t\to\infty}\left(-\frac{1}{\ln t} + \frac{1}{\ln 3}\right) = \frac{1}{\ln 3}$$

$\therefore$ THE INTEGRAL CONVERGES

$\therefore \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ CONVERGES BY THE INTEGRAL TEST

ii) NOTICE THAT

$$\frac{1}{\sqrt{n^2-n}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

NOW, $\sum_{n=2}^{\infty} \frac{1}{n}$ DIVERGES (p-SERIES, $p=1$)

$\therefore \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-n}}$ DIVERGES BY COMPARISON TEST

iii) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{\cancel{2^n}}{\cancel{2^n}\cdot 2} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{1}{2} = \frac{1}{2} < 1$

$\therefore \sum_{n=1}^{\infty} \frac{n^2}{2^n}$ CONVERGES BY RATIO TEST

11) $f(x) = \cos(3x)$ $\quad f(0) = 1$

$f'(x) = -3\sin(3x)$ $\quad f'(0) = 0$

$f''(x) = -9\cos(3x)$ $\quad f''(0) = -9$

$f'''(x) = 27\sin(3x)$ $\quad f'''(0) = 0$

$f^{(4)}(x) = 81\cos(3x)$ $\quad f^{(4)}(0) = 81$

$\therefore T_4(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3$

$+ \frac{f^{(4)}(0)}{4!}(x-0)^4$

$= 1 - \frac{9}{2}x^2 + \frac{81}{24}x^4$

$= 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4$