EXERCISES Use the definition of definite integral (Riemann Sum) to evaluate each of the following definite integrals. NOTE: these functions are not necessarily non-negative. Nevertheless, the definition (Riemann Sum) may still be applied (as shown in the solutions).

1. $\int_0^5 4x\,dx$

2. $\int_0^2 \left(x^2+10\right)dx$

3. $\int_0^3 \left(5x^2-2x\right)dx$

4. $\int_0^4 \left(-3x^2+5x-1\right)dx$

5. $\int_4^6 2x^2\,dx$

6. $\int_1^5 \left(x-4x^2\right)dx$

7. $\int_{-2}^0 \left(3x^2+2x\right)dx$

8. $\int_{-3}^0 \left(4x^2-5x-1\right)dx$

9. $\int_{-8}^{-3} \left(4-2x^2\right)dx$

10. $\int_{-5}^{-1} \left(x^2+3x+5\right)dx$

11. $\int_{-2}^3 \left(1-5x^2\right)dx$

12. $\int_{-1}^2 x^3\,dx$

SOLUTIONS

1. $\Delta x=\dfrac{5}{n} \;\rightarrow\; x_k=\dfrac{5k}{n}$ and $f(x_k)=4\left(\dfrac{5k}{n}\right) \;\rightarrow\; f(x_k)=\dfrac{20k}{n}$

$$f(x_k)\Delta x = \left(\frac{20k}{n}\right)\left(\frac{5}{n}\right) = \frac{100k}{n^2}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{100k}{n^2}\right) = \frac{100}{n^2}\sum_{k=1}^{n} k = \frac{100}{2}\left(1+\frac{1}{n}\right)$$

$$\int_0^5 4x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\left[50\left(1+\frac{1}{n}\right)\right] = 50$$

2. $\Delta x = \frac{2}{n} \rightarrow x_k = \frac{2k}{n}$ and $f(x_k) = \left(\frac{2k}{n}\right)^2 + 10 \rightarrow f(x_k) = \frac{4k^2}{n^2} + 10$

$$f(x_k)\Delta x = \left(\frac{4k^2}{n^2} + 10\right)\left(\frac{2}{n}\right) = \frac{8k^2}{n^3} + \frac{20}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{8k^2}{n^3} + \frac{20}{n}\right) = \frac{8}{n^3}\sum_{k=1}^{n} k^2 + \frac{20}{n}\sum_{k=1}^{n} 1 = \frac{8}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 20$$

$$\int_0^2 \left(x^2 + 10\right) dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\left[\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 20\right] = \frac{4}{3}(1)(2) + 20 = \frac{68}{3}$$

3. $\Delta x = \frac{3}{n} \rightarrow x_k = \frac{3k}{n}$ and $f(x_k) = 5\left(\frac{3k}{n}\right)^2 - 2\left(\frac{3k}{n}\right) \rightarrow f(x_k) = \frac{45k^2}{n^2} - \frac{6k}{n}$

$$f(x_k)\Delta x = \left(\frac{45k^2}{n^2} - \frac{6k}{n}\right)\left(\frac{3}{n}\right) = \frac{135k^2}{n^3} - \frac{18k}{n^2}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{135}{n^3}\sum_{k=1}^{n} k^2 - \frac{18}{n^2}\sum_{k=1}^{n} k = \frac{135}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{18}{2}\left(1+\frac{1}{n}\right)$$

$$\int_0^3 \left(5x^2 - 2x\right) dx = \lim_{n\to\infty}\left[\frac{45}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 9\left(1+\frac{1}{n}\right)\right] = \frac{45}{2}(1)(2) - 9(1) = 36$$

4. $\Delta x = \frac{4}{n} \rightarrow x_k = \frac{4k}{n}$ and $f(x_k) = -3\left(\frac{4k}{n}\right)^2 + 5\left(\frac{4k}{n}\right) - 1 \rightarrow f(x_k) = -\frac{48k^2}{n^2} + \frac{20k}{n} - 1$

$$f(x_k)\Delta x = \left(-\frac{48k^2}{n^2} + \frac{20k}{n} - 1\right)\left(\frac{4}{n}\right) = -\frac{192k^2}{n^3} + \frac{80k}{n^2} - \frac{4}{n}$$

$$\sum_{i=1}^{n} f(x_k)\Delta x = -\frac{192}{n^3}\sum_{i=1}^{n} k^2 + \frac{80}{n^2}\sum_{i=1}^{n} k - \frac{4}{n}\sum_{i=1}^{n} 1 = -\frac{192}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{80}{2}\left(1+\frac{1}{n}\right) - 4$$

$$\int_0^4 \left(-3x^2 + 5x - 1\right) dx = \lim_{n\to\infty}\left[-32\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 40\left(1+\frac{1}{n}\right) - 4\right] = -32(1)(2) + 40(1) - 4 = -28$$

5. $\Delta x = \frac{2}{n} \rightarrow x_k = 4 + \frac{2k}{n}$ and $f(x_k) = 2\left(4 + \frac{2k}{n}\right)^2 \rightarrow f(x_k) = \frac{8k^2}{n^2} + \frac{32k}{n} + 32$

$$f(x_k)\Delta x = \left(\frac{8k^2}{n^2} + \frac{32k}{n} + 32\right)\left(\frac{2}{n}\right) = \frac{16k^2}{n^3} + \frac{64k}{n^2} + \frac{64}{n}$$

$$\sum_{i=1}^{n} f(x_k)\Delta x = \frac{16}{n^3}\sum_{i=1}^{n} k^2 + \frac{64}{n^2}\sum_{i=1}^{n} k + \frac{64}{n}\sum_{i=1}^{n} 1 = \frac{16}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \frac{64}{2}\left(1+\frac{1}{n}\right) + 64$$

$$\int_4^6 2x^2\, dx = \lim_{n\to\infty}\left[\frac{8}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 64\left(1+\frac{1}{n}\right)\right] = \frac{8}{3}(1)(2) + 32(1) + 64 = \frac{304}{3}$$

6. $\Delta x=\frac{4}{n} \to x_k=1+\frac{4k}{n}$ and $f(x_k)=\left(1+\frac{4k}{n}\right)-4\left(1+\frac{4k}{n}\right)^2 \to f(x_k)=-\frac{64k^2}{n^2}-\frac{28k}{n}-3$

$$f(x_k)\Delta x=\left(-\frac{64k^2}{n^2}-\frac{28k}{n}-3\right)\left(\frac{4}{n}\right)=-\frac{256k^2}{n^3}-\frac{112k}{n^2}-\frac{12}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x=-\frac{256}{n^3}\sum_{k=1}^{n}k^2-\frac{112}{n^2}\sum_{k=1}^{n}k-\frac{12}{n}\sum_{k=1}^{n}1=-\frac{256}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{112}{2}\left(1+\frac{1}{n}\right)-12$$

$$\int_1^5 (x-4x^2)dx=\lim_{n\to\infty}\left[-\frac{128}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-56\left(1+\frac{1}{n}\right)-12\right]=-\frac{128}{3}(1)(2)-56(1)-12=-\frac{460}{3}$$

7. $\Delta x=\frac{2}{n} \to x_k=\frac{2k}{n}-2$ and $f(x_k)=3\left(\frac{2k}{n}-2\right)^2+2\left(\frac{2k}{n}-2\right) \to f(x_k)=\frac{12k^2}{n^2}-\frac{20k}{n}+8$

$$f(x_k)\Delta x=\left(\frac{12k^2}{n^2}-\frac{20k}{n}+8\right)\left(\frac{2}{n}\right)=\frac{24k^2}{n^3}-\frac{40k}{n^2}+\frac{16}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x=\frac{24}{n^3}\sum_{k=1}^{n}k^2-\frac{40}{n^2}\sum_{k=1}^{n}k+\frac{16}{n}\sum_{k=1}^{n}1=\frac{24}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{40}{2}\left(1+\frac{1}{n}\right)+16$$

$$\int_{-2}^{0}(3x^2+2x)\,dx=\lim_{n\to\infty}\left[4\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-20\left(1+\frac{1}{n}\right)+16\right]=4(1)(2)-20(1)+16=4$$

8. $\Delta x=\frac{3}{n} \to x_k=\frac{3k}{n}-3$ and $f(x_k)=4\left(\frac{3k}{n}-3\right)^2-5\left(\frac{3k}{n}-3\right)-1 \to f(x_k)=\frac{36k^2}{n^2}-\frac{87k}{n}+50$

$$f(x_k)\Delta x=\left(\frac{36k^2}{n^2}-\frac{87k}{n}+50\right)\left(\frac{3}{n}\right)=\frac{108k^2}{n^3}-\frac{261k}{n^2}+\frac{150}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x=\frac{108}{n^3}\sum_{k=1}^{n}k^2-\frac{261}{n^2}\sum_{k=1}^{n}k+\frac{150}{n}\sum_{k=1}^{n}1=\frac{108}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{261}{2}\left(1+\frac{1}{n}\right)+150$$

$$\int_{-3}^{0}(4x^2-5x-1)dx=\lim_{n\to\infty}\left[18\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{261}{2}\left(1+\frac{1}{n}\right)+150\right]=18(1)(2)-\frac{261}{2}(1)+150=\frac{111}{2}$$

9. $\Delta x=\frac{5}{n} \to x_k=\frac{5k}{n}-8$ and $f(x_k)=4-2\left(\frac{5k}{n}-8\right)^2 \to f(x_k)=-\frac{50k^2}{n^2}+\frac{160k}{n}-124$

$$f(x_k)\Delta x=\left(-\frac{50k^2}{n^2}+\frac{160k}{n}-124\right)\left(\frac{5}{n}\right)=-\frac{250k^2}{n^3}+\frac{800k}{n^2}-\frac{620}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x=-\frac{250}{n^3}\sum_{k=1}^{n}k^2+\frac{800}{n^2}\sum_{k=1}^{n}k-\frac{620}{n}\sum_{k=1}^{n}1=-\frac{250}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{800}{2}\left(1+\frac{1}{n}\right)-620$$

$$\int_{-8}^{-3}(4-2x^2)dx=\lim_{n\to\infty}\left[-\frac{125}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+400\left(1+\frac{1}{n}\right)-620\right]=-\frac{125}{3}(1)(2)+400(1)-620=\frac{304}{3}$$

10. $\Delta x=\frac{4}{n} \to x_k=\frac{4k}{n}-5$ and $f(x_k)=\left(\frac{4k}{n}-5\right)^2+3\left(\frac{4k}{n}-5\right)+5 \to f(x_k)=\frac{16k^2}{n^2}-\frac{28k}{n}+15$

$$f(x_k)\Delta x = \left(\frac{16k^2}{n^2}-\frac{28k}{n}+15\right)\left(\frac{4}{n}\right) = \frac{64k^2}{n^3}-\frac{112k}{n^2}+\frac{60}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{64}{n^3}\sum_{k=1}^{n} k^2 - \frac{112}{n^2}\sum_{k=1}^{n} k + \frac{60}{n}\sum_{k\to 1}^{n} 1 = \frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-56\left(1+\frac{1}{n}\right)+60$$

$$\int_{-5}^{-1}\left(x^2+3x+5\right)dx = \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-56\left(1+\frac{1}{n}\right)+60\right] = \frac{32}{3}(1)(2)-56(1)+60 = \frac{76}{3}$$

11. $\Delta x=\frac{5}{n} \to x_k=\frac{5k}{n}-2$ and $f(x_k)=1-5\left(\frac{5k}{n}-2\right)^2 \to f(x_k)=-\frac{125k^2}{n^2}+\frac{100k}{n}-19$

$$f(x_k)\Delta x = \left(-\frac{125k^2}{n^2}+\frac{100k}{n}-19\right)\left(\frac{5}{n}\right) = -\frac{625k^2}{n^3}+\frac{500k}{n^2}-\frac{95}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = -\frac{625}{n^3}\sum_{k=1}^{n} k^2+\frac{500}{n^2}\sum_{k=1}^{n} k-\frac{95}{n}\sum_{k=1}^{n} 1 = -\frac{625}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+\frac{500}{2}\left(1+\frac{1}{n}\right)-95$$

$$\int_{-2}^{3}\left(1-5x^2\right)dx = \lim_{n\to\infty}\left[-\frac{625}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+250\left(1+\frac{1}{n}\right)-95\right] = -\frac{625}{6}(1)(2)+250(1)-95 = -\frac{160}{3}$$

12. $\Delta x=\frac{3}{n} \to x_k=\frac{3k}{n}-1$ and $f(x_k)=\left(\frac{3k}{n}-1\right)^3 \to f(x_k)=\frac{27k^3}{n^3}-\frac{27k^2}{n^2}+\frac{9k}{n}-1$

$$f(x_k)\Delta x = \left(\frac{27k^3}{n^3}-\frac{27k^2}{n^2}+\frac{9k}{n}-1\right)\left(\frac{3}{n}\right) = \frac{81k^3}{n^4}-\frac{81k^2}{n^3}+\frac{27k}{n^2}-\frac{3}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n}\left(\frac{81k^3}{n^4}-\frac{81k^2}{n^3}+\frac{27k}{n^2}-\frac{3}{n}\right) = \frac{81}{n^4}\sum_{k=1}^{n} k^3-\frac{81}{n^3}\sum_{k=1}^{n} k^2+\frac{27}{n^2}\sum_{k=1}^{n} k-\frac{3}{n}\sum_{k=1}^{n} 1$$

$$= \frac{81}{n^4}\cdot\frac{n^2(n+1)^2}{4} - \frac{81}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{27}{n^2}\cdot\frac{n(n+1)}{2} - \frac{3}{n}\cdot n$$

$$= \frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{27}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{27}{2}\left(1+\frac{1}{n}\right) - 3$$

$$\int_{-1}^{2} x^3\,dx = \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - \frac{27}{2}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{27}{2}\left(1+\frac{1}{n}\right) - 3\right]$$

$$= \left[\frac{81}{4}(1)^2 - \frac{27}{2}(1)(2) + \frac{27}{2}(1) - 3\right] = \frac{15}{4}$$