Dawson College: **Calculus II**: 201-NYB-05 12

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Last Name: SOLUTIONS

First Name: ______

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Test 1

Question 1. *(5 marks)* Evaluate the following indefinite integrals:

(a) $\int \frac{1}{x} dx = \ln|x| + c$

(b) $\int \sec x dx = \ln|\sec x + \tan x| + C$

(c) $\int \sec x \tan x dx = \sec x + c$

(d) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

(e) $\int \sec^2 x dx = \tan x + C$

Question 2. *(8 marks)* Given:

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Evaluate the following integral using the definition of the definite integral.

$$\int_1^3 (3x^2 - 4x)\,dx \qquad \circ\ \Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$\circ\ x_i = a + i\Delta x = 1 + \frac{2i}{n}$

$\circ\ f(x_i) = 3x_i^2 - 4x_i = 3\left(1+\frac{2i}{n}\right)^2 - 4\left(1+\frac{2i}{n}\right) =$

$= 3\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - 4 - \frac{8i}{n} = 3 + \frac{12i}{n} + \frac{12i^2}{n^2} - 4 - \frac{8i}{n}$

$= -1 + \frac{4i}{n} + \frac{12i^2}{n^2}$

$\circ\ f(x_i)\Delta x = \left(-1 + \frac{4i}{n} + \frac{12i^2}{n^2}\right)\left(\frac{2}{n}\right) = -\frac{2}{n} + \frac{8i}{n^2} + \frac{24i^2}{n^3}$

$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(-\frac{2}{n} + \frac{8i}{n^2} + \frac{24i^2}{n^3}\right) = -\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{24}{n^3}\sum_{i=1}^{n} i^2$

$= -\frac{2}{n}\cdot n + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{24}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$

$= -2 + 4\cdot 1\cdot(1+1/n) + 4\cdot 1\cdot(1+1/n)\cdot(2+1/n)$

$\therefore \int_1^3 (3x^2-4x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$

$= \lim_{n\to\infty}\left[-2 + 4(1+1/n) + 4(1+1/n)\cdot(2+1/n)\right]$

$= -2 + 4(1+0) + 4(1+0)(2+0)$

$= -2 + 4 + 8$

$= 10$

Question 4. *(5 marks)* Find the average value of the function $f(x) = \cos^4 x \sin x$ on the interval $[0, \pi]$.

AVERAGE VALUE $= \dfrac{1}{\pi - 0} \displaystyle\int_0^{\pi} \cos^4 x \sin x \, dx$

LET $u = \cos x$

$du = -\sin x \, dx$

IF $x = 0 \Rightarrow u = \cos(0) = 1$

IF $x = \pi \Rightarrow u = \cos(\pi) = -1$

$= \dfrac{1}{\pi} \displaystyle\int_1^{-1} u^4 (-1) \, du$

$= -\dfrac{1}{\pi} \displaystyle\int_1^{-1} u^4 \, du = -\dfrac{1}{\pi} \left[\dfrac{u^5}{5} \right]_1^{-1} = -\dfrac{1}{\pi} \left[\dfrac{(-1)^5}{5} - \dfrac{1^5}{5} \right]$

$= -\dfrac{1}{\pi} \left[\dfrac{-1}{5} - \dfrac{1}{5} \right] = \dfrac{2}{5\pi}$

Question 5. Evaluate the following integrals. Simplify as much as possible.

(a) *(4 marks)* $\displaystyle\int_1^4 \sqrt{t} \ln t \, dt =$

LET

$u = \ln t \qquad dv = t^{1/2} \, dt$

$du = \dfrac{1}{t} dt \qquad v = \dfrac{2}{3} t^{3/2}$

$= uv\Big|_1^4 - \displaystyle\int_1^4 v \, du =$

$= \left[\dfrac{2}{3} t^{3/2} \ln t \right]_1^4 - \displaystyle\int_1^4 \dfrac{2}{3} t^{3/2} \cdot \dfrac{1}{t} \, dt = \dfrac{2}{3} \left[t^{3/2} \ln t \right]_1^4 - \dfrac{2}{3} \displaystyle\int_1^4 t^{1/2} \, dt$

$= \dfrac{2}{3} \left[8 \ln 4 - 1 \ln 1 \right] - \dfrac{2}{3} \left[\dfrac{2}{3} t^{3/2} \right]_1^4$

$= \dfrac{2}{3} \left[8 \ln 4 - 0 \right] - \dfrac{4}{9} \left[8 - 1 \right]$

$= \dfrac{16}{3} \ln 4 - \dfrac{28}{9}$

(b) *(4 marks)* $\int_1^4 \frac{1+x^{3/2}+x^2}{x^{5/2}}\,dx \quad = \int_1^4 \left(x^{-5/2} + x^{-1} + x^{-1/2}\right) dx$

$= \left[-\frac{2}{3} x^{-3/2} + \ln|x| + 2x^{1/2} \right]_1^4$

$= \left[-\frac{2}{3} \cdot \frac{1}{(4)^{3/2}} + \ln 4 + 2(4)^{1/2} \right] - \left[-\frac{2}{3} \cdot \frac{1}{(1)^{3/2}} + \ln 1 + 2(1)^{1/2} \right]$

$= -\frac{2}{3} \cdot \frac{1}{8} + \ln 4 + 4 + \frac{2}{3} - 0 - 2$

$= -\frac{2}{24} + \ln 4 + \frac{2}{3} + 2 \quad = \quad \frac{31}{12} + \ln 4$

(c) *(4 marks)* $\int \frac{x}{\sqrt{x+5}}\,dx$

LET $u = x+5$
$du = dx$
AND $x = u-5$

$= \int \frac{u-5}{\sqrt{u}}\,du = \int \frac{u-5}{u^{1/2}}\,du$

$= \int \left(u^{1/2} - 5u^{-1/2}\right) du = \frac{2}{3}u^{3/2} - 10u^{1/2} + C$

$= \frac{2}{3}(x+5)^{3/2} - 10(x+5)^{1/2} + C$

(d) *(4 marks)* $\int x^2 e^{2x}\,dx$

LET $u = x^2$ $\quad dv = e^{2x}dx$

$du = 2x\,dx$ $\quad v = \frac{1}{2}e^{2x}$

$= uv - \int v\,du = \frac{1}{2}x^2 e^{2x} - \int x e^{2x}\,dx$

LET $u = x$ $\quad dv = e^{2x}dx$

$du = dx$ $\quad v = \frac{1}{2}e^{2x}$

$= \frac{1}{2}x^2 e^{2x} - \left[uv - \int v\,du\right]$

$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{2}\int e^{2x}\,dx$

$= \frac{1}{2}x^2 e^{2x} - \frac{1}{2}x e^{2x} + \frac{1}{4}e^{2x} + C$

(e) *(3 marks)* $\int_{-2}^{2} \frac{x^5 - 3x^3 + 2x}{x^6 + 4x^2 + 6}\,dx$

LET $f(x) = \dfrac{x^5 - 3x^3 + 2x}{x^6 + 4x^2 + 6}$

$f(-x) = \dfrac{(-x)^5 - 3(-x)^3 + 2(-x)}{(-x)^6 + 4(-x)^2 + 6}$

$= \dfrac{-x^5 + 3x^3 - 2}{x^6 + 4x^2 + 6}$

$= -\dfrac{x^5 - 3x^3 + 2}{x^6 + 4x^2 + 6} = -f(x)$

$\therefore$ f IS ODD

$\therefore \int_{-2}^{2} \dfrac{x^5 - 3x^3 + 2x}{x^6 + 4x^2 + 6}\,dx = 0$

(f) *(5 marks)* $\int e^{\theta} \sin\theta \, d\theta$

LET $u = e^{\theta}$ $\quad dv = \sin\theta \, d\theta$

$du = e^{\theta} d\theta$ $\quad v = -\cos\theta$

$= uv - \int v \, du$

$= -e^{\theta}\cos\theta + \int e^{\theta} \cos\theta \, d\theta$

LET $u = e^{\theta}$ $\quad dv = \cos\theta \, d\theta$

$du = e^{\theta} d\theta$ $\quad v = \sin\theta$

$= -e^{\theta}\cos\theta + \left[uv - \int v \, du \right]$

$= -e^{\theta}\cos\theta + e^{\theta}\sin\theta - \int e^{\theta} \sin\theta$

$\Rightarrow 2\int e^{\theta} \sin\theta \, d\theta = -e^{\theta}\cos\theta + e^{\theta}\sin\theta + C_0$

$\therefore \int e^{\theta} \sin\theta \, d\theta = \dfrac{-e^{\theta}\cos\theta + e^{\theta}\sin\theta}{2} + C_1$

Question 6. *(5 marks)* If f is continuous and $\int_0^3 f(x)\,dx = 12$ find

$$\int_{-1}^{2} x f(x^2-1)\,dx$$

$$= \int_0^3 x f(u) \frac{du}{2x}$$

$$= \frac{1}{2}\int_0^3 f(u)\,du$$

$$= \frac{1}{2}(12)$$

$$= 6$$

LET $u = x^2 - 1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

IF $x = -1 \Rightarrow u = (-1)^2 - 1$
$= 0$

$x = 2 \Rightarrow u = 2^2 - 1$
$= 3$

Question 7. *(5 marks)* Find the derivative of the following function. Clearly show all of your work for full marks.

$$G(x) = \int_{\cos^2 x}^{2} \sqrt{1+t^3}\,dt = -\int_2^{\cos^2 x} \sqrt{1+t^3}\,dt = -f(h(x))$$

WHERE $f(x) = \int_2^{x} \sqrt{1+t^3}\,dt \Rightarrow f'(x) = \sqrt{1+x^3}$

$h(x) = \cos^2 x \Rightarrow h'(x) = -2\cos x \sin x$

$\therefore G'(x) = f'(h(x)) \cdot h'(x)$

$= -\sqrt{1+(\cos^2 x)^3} \cdot (-2\cos x \sin x)$

$= 2\cos x \sin x \sqrt{1+\cos^6 x}$

Bonus. *(3 marks)* Show that

$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$

$\int \csc x\,dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\,dx$

LET
$u = \csc x + \cot x$
$du = (-\csc x \cot x - \csc^2 x)\,dx$

$= \int \frac{\cancel{\csc^2 x + \csc x \cot x}}{u} \frac{du}{-(\csc^2 x + \csc x \cot x)}$

$= -\int \frac{1}{u}\,du = -\ln|u| + C$

$= -\ln|\csc x + \cot x| + C$