Dawson College: **Calculus II**: 201-NYB-05 12

May 5, 2010

Last Name: SOLUTIONS

First Name: ______

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Test 3

Question 1. *(5 marks)* Evaluate the following improper integral:

$$\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx = \underbrace{\int_{-\infty}^{0} x^2 e^{-x^3}\,dx}_{I_1} + \underbrace{\int_{0}^{\infty} x^2 e^{-x^3}\,dx}_{I_2}$$

$$\int x^2 e^{-x^3}\,dx = \int \cancel{x^2} e^u \frac{du}{-3\cancel{x^2}}$$

LET $u = -x^3$

$du = -3x^2\,dx$

$dx = \frac{du}{-3x^2}$

$$= -\frac{1}{3}\int e^u\,du = -\frac{1}{3}e^u + c$$

$$= -\frac{1}{3}e^{-x^3} + c$$

$$I_1 = \lim_{t\to-\infty}\int_t^0 x^2 e^{-x^3}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0$$

$$= \lim_{t\to-\infty}\left[-\frac{1}{3}e^0 + \frac{1}{3}e^{-t^3}\right]$$

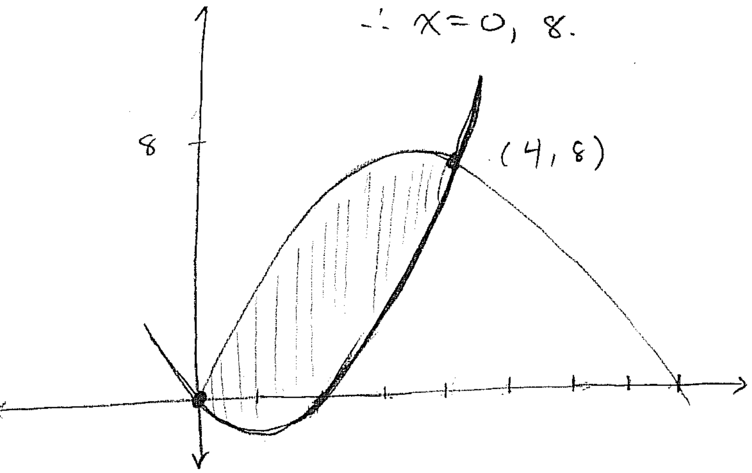
$$= \infty$$

$\therefore$ THE INTEGRAL DIVERGES.

Question 2. *(5 marks)* Find the area between the curves $y = x^2 - 2x$ and $y = 6x - x^2$

INTERSECTION: $x^2 - 2x = 6x - x^2 \Rightarrow 2x^2 - 8x = 0 \Rightarrow x(x-8) = 0$

$\therefore x = 0, 8.$



$\therefore$ AREA $= \int_0^4 (6x - x^2) - (x^2 - 2x)\, dx$

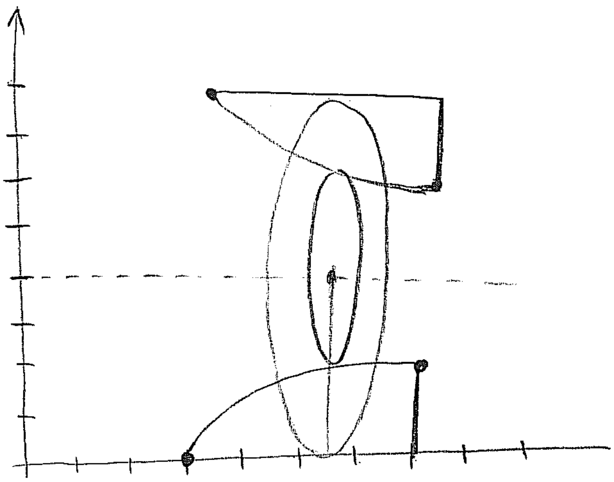
$= \int_0^4 8x - 2x^2\, dx$

$= \left[4x^2 - \frac{2}{3} x^3 \right]_0^4$

$= 4(4)^2 - \frac{2}{3}(4)^3$

$= \frac{64}{3}$ units^2

Question 3. *(10 marks)* Find the volume of the region bounded by $y = \sqrt{x-3}$, $x = 7$ and $y = 0$ about $y = 4$ using :
(a) the disc/washer method.



$A(x) = \pi R^2 - \pi r^2$, $R = 4$, $r = 4 - \sqrt{x-3}$

$= \pi(4)^2 - \pi(4 - \sqrt{x-3})^2$

$= 16\pi - \pi(16 - 8\sqrt{x-3} + (x-3))$

$= 8\pi\sqrt{x-3} - \pi x + 3\pi$

$V = \int_3^7 (8\pi\sqrt{x-3} - \pi x + 3\pi)\,dx$

$= \left[8\pi \frac{(x-3)^{3/2}}{3/2} - \pi\frac{x^2}{2} + 3\pi x\right]_3^7$

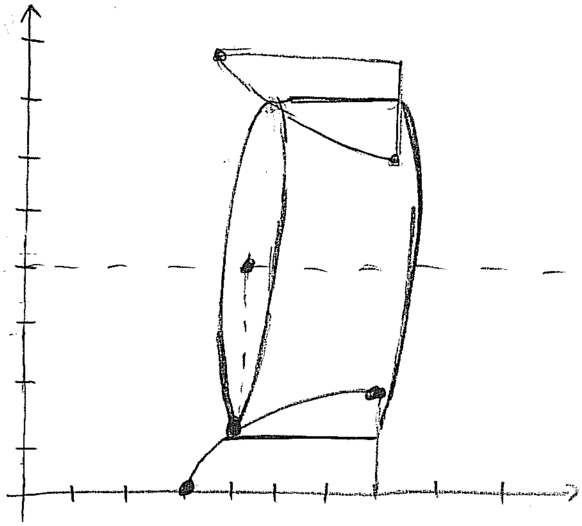
$= \frac{16}{3}\pi(7-3)^{3/2} - \frac{\pi}{2}(7)^2 + 3\pi(7) - \left[\frac{16}{3}(0)^{3/2} - \frac{\pi}{2}(3)^2 + 3\pi(3)\right]$

$= \frac{16\pi(8)}{3} - \frac{49\pi}{2} + 21\pi + \frac{9}{2}\pi - 9\pi$

$= \frac{128}{3}\pi - \frac{40}{2}\pi + 12\pi$

$= \frac{104}{3}\pi$ units^3

(b) the method of cylindrical shells.



$A(y) = 2\pi r h$

$r = 4 - y$

$h = 7 - x$

$= 7 - (y^2 + 3)$

$= 4 - y^2$

$\therefore A(y) = 2\pi(4-y)(4-y^2)$

$= 2\pi(16 - 4y - 4y^2 + y^3)$

$\therefore V = \int_0^2 2\pi(16 - 4y - 4y^2 + y^3)\,dy$

$= 2\pi\left[16y - 2y^2 - \frac{4}{3}y^3 + \frac{y^4}{4}\right]_0^2$

$= 2\pi\left[16(2) - 2(2)^2 - \frac{4}{3}(2)^3 + \frac{(2)^4}{4} - 0\right]$

$= 2\pi\left[32 - 8 - \frac{32}{3} + 4\right]$

$= 2\pi\left[28 - \frac{32}{3}\right]$

$= \frac{104}{3}\pi$ units^3

$3(y-1)^2 = 2(x+1)^3$

Question 4. *(5 marks)* Find the length of the curve ~~$2y^2 = (3x-2)^3$~~ from $x = \cancel{1}\ 0$ to $x = 2$.

$$y = 1 + \sqrt{\tfrac{2}{3}}(x+1)^{3/2}, \quad y' = \sqrt{\tfrac{2}{3}} \cdot \tfrac{3}{2}(x+1)^{1/2}$$

$$\therefore \quad L = \int_0^2 \sqrt{1 + \left(\sqrt{\tfrac{2}{3}} \cdot \tfrac{3}{2}(x+1)^{1/2}\right)^2}\, dx$$

$$= \int_0^2 \sqrt{1 + \tfrac{2}{\cancel{3}} \cdot \tfrac{\cancel{9}\,3}{\cancel{4}\,2}(x+1)}\, dx$$

$$= \int_0^2 \sqrt{1 + \tfrac{3}{2}(x+1)}\, dx$$

$$= \int_0^2 \sqrt{\tfrac{3}{2}x + \tfrac{5}{2}}\, dx$$

$$= \left[\frac{2}{3} \cdot \frac{\left(\frac{3}{2}x + \frac{5}{2}\right)^{3/2}}{3/2}\right]_0^2 = \left[\frac{4}{9}\left(\frac{3}{2}x + \frac{5}{2}\right)^{3/2}\right]_0^2$$

$$= \frac{4}{9}\left(\frac{3}{2}(2) + \frac{5}{2}\right)^{3/2} - \frac{4}{9}\left(0 + \frac{5}{2}\right)^{3/2}$$

$$= \frac{4}{9}\left(\frac{11}{2}\right)^{3/2} - \frac{4}{9}\left(\frac{5}{2}\right)^{3/2}$$

Question 5. *(8 marks)* Determine whether the following sequences are convergent. If it is convergent find what it converges to.

(a) $a_n = \dfrac{(n+1)!}{(n+3)!\,n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(n+1)!}{(n+3)!\,n} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}}{\cancel{(n+1)!}\,(n+2)(n+3)n}$$

$$= \lim_{n\to\infty} \frac{1}{(n+2)(n+3)n} = 0$$

(b) $a_n = \dfrac{(-1)^n \ln n}{n}$

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \lim_{n\to\infty} \frac{\ln n}{n} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0$$

$$\therefore \lim_{n\to\infty} a_n = 0$$

Question 6. *(17 marks)* Determine whether the following series are convergent. If it is convergent find what it converges to. Make sure to state any theorems or tests that you use.

(a) $\sum_{n=\phi}^{\infty} \frac{3+2\cdot 5^{n-1}}{6^{n-1}} = \sum_{n=\phi}^{\infty} 3\left(\frac{1}{6}\right)^{n-1} + \sum_{n=\phi}^{\infty} \frac{2\cdot 5^{n-1}}{6^{n-1}}$

$= \sum_{n=\phi}^{\infty} 3\left(\frac{1}{6}\right)^{n-1} + \sum_{n=\phi}^{\infty} 2\left(\frac{5}{6}\right)^{n-1}$

GEOMETRIC SERIES $|r| = \frac{1}{6} < 1$ $\qquad |r| = \frac{5}{6} < 1$

$= \frac{3}{1-1/6} + \frac{2}{1-5/6} = \frac{3}{5/6} + \frac{2}{1/6} = \frac{18}{5} + 12$

$= \frac{78}{5}$

(b) $\sum_{n=0}^{\infty} \frac{5n^2+3n-1}{4n^2+2}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^2+3n-1}{4n^2+2} = \lim_{n\to\infty} \frac{5+3/n-1/n^2}{4+2/n^2}$

$= \frac{5+0-0}{4+0} = \frac{5}{4} \neq 0$

$\therefore$ THE SERIES DIVERGES BY TEST FOR DIVERGENCE.

(c) $\sum_{n=0}^{\infty} 5\cdot\left(\frac{1}{3}\right)^{n+1} = \sum_{n=1}^{\infty} 5\cdot\left(\frac{1}{3}\right)^{n}$

$$= \sum_{n=1}^{\infty} \frac{5}{3}\cdot\left(\frac{1}{3}\right)^{n-1} = \frac{5/3}{1-1/3} = \frac{5/3}{2/3}$$

GEOMETRIC SERIES
$|r| = \frac{1}{3}$

$$= \frac{5}{3}\cdot\frac{3}{2} = \frac{5}{2}$$

$\therefore$ THE SERIES CONVERGES.

(d) $\sum_{n=2}^{\infty}\left(\frac{2}{n}-\frac{2}{n+2}\right)$

$$S_n = \left(\frac{2}{2}-\cancel{\frac{2}{4}}\right)+\left(\frac{2}{3}-\cancel{\frac{2}{5}}\right)+\left(\cancel{\frac{2}{4}}-\cancel{\frac{2}{6}}\right)+\left(\cancel{\frac{2}{5}}-\cancel{\frac{2}{7}}\right)$$
$$+\left(\cancel{\frac{2}{n-2}}-\cancel{\frac{2}{n}}\right)+\left(\cancel{\frac{2}{n-1}}-\frac{2}{n+1}\right)+\left(\cancel{\frac{2}{n}}-\frac{2}{n+2}\right)$$

$$= 1+\frac{2}{3}-\frac{2}{n+1}-\frac{2}{n+2}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{2}{3}-\frac{2}{n+1}-\frac{2}{n+2}\right) = 1+\frac{2}{3} = \frac{5}{3}$$

$\therefore$ THE SERIES CONVERGES AND $\sum_{n=2}^{\infty}\left(\frac{2}{n}-\frac{2}{n+2}\right) = \frac{5}{3}$

Bonus. *(3 marks)* Achilles and a tortoise have entered into a 1000m foot race. To make the race more fair, Achilles gives the tortoise a 400m head start. By the time Achilles gets 400m into the race the tortoise has gone 200m. When Achilles goes another 200m the tortoise has gone another 100m. When Achilles goes another 100m the tortoise has gone another 50m. Does Achilles catch the tortoise before the end of the race?

TO CATCH THE TORTOISE ACHILLES MUST RUN

$$400 + 200 + 100 + 50 + 25 + \ldots$$

$$= 400 + 400\left(\tfrac{1}{2}\right) + 400\left(\tfrac{1}{2}\right)^2 + 400\left(\tfrac{1}{2}\right)^3 + \ldots$$

$$= \sum_{n=0}^{\infty} 400\left(\tfrac{1}{2}\right)^{n-1} = \frac{400}{1-\frac{1}{2}} = \frac{400}{\frac{1}{2}} = 800 < 1000$$

GEOMETRIC SERIES WITH $|r| = \frac{1}{2} < 1$

$\therefore$ CONVERGES

$\therefore$ ACHILLES WILL CATCH THE TORTOISE AT 800m,

IE. BEFORE THE END OF THE RACE.