Dawson College: Linear Algebra: 201-NYC-05 06 February 24, 2010

Last Name: SOLUTIONS

First Name: ____________

Student ID: ____________

Test 1

Question 1. *(8 marks)* Solve the following system using Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcrcr} & & 2x_2 & + & 4x_3 & + & 6x_4 & - & 5x_5 & = & -3 \\ 2x_1 & + & 4x_2 & + & 14x_3 & & & - & 2x_5 & = & 12 \\ 3x_1 & + & 6x_2 & + & 21x_3 & - & 6x_4 & - & 6x_5 & = & -6 \end{array}$$

AUGMENTED MATRIX:

$$\left[\begin{array}{ccccc|c} 0 & 2 & 4 & 6 & -5 & -3 \\ 2 & 4 & 14 & 0 & -2 & 12 \\ 3 & 6 & 21 & -6 & -6 & -6 \end{array}\right] R_1 \Leftrightarrow R_2 \left[\begin{array}{ccccc|c} 2 & 4 & 14 & 0 & -2 & 12 \\ 0 & 2 & 4 & 6 & -5 & -3 \\ 3 & 6 & 21 & -6 & -6 & -6 \end{array}\right] \xrightarrow{R_1 \cdot 1/2}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 7 & 0 & -1 & 6 \\ 0 & 2 & 4 & 6 & -5 & -3 \\ 3 & 6 & 21 & -6 & -6 & -6 \end{array}\right] \xrightarrow{R_3 - 3R_1} \left[\begin{array}{ccccc|c} 1 & 2 & 7 & 0 & -1 & 6 \\ 0 & 2 & 4 & 6 & -5 & -3 \\ 0 & 0 & 0 & -6 & -3 & -24 \end{array}\right] \xrightarrow{R_2 + R_3}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 7 & 0 & -1 & 6 \\ 0 & 2 & 4 & 0 & -8 & -27 \\ 0 & 0 & 0 & -6 & -3 & -24 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 7 & 33 \\ 0 & 2 & 4 & 0 & -8 & -27 \\ 0 & 0 & 0 & -6 & -3 & -24 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 \cdot 1/2} \\ \xrightarrow{R_3 \cdot (-1/6)} \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 7 & 33 \\ 0 & 1 & 2 & 0 & -4 & -27/2 \\ 0 & 0 & 0 & 1 & 1/2 & 4 \end{array}\right]$$

FREE VARIABLES:
LET $x_3 = s$, $x_5 = t$

- $x_1 + 3x_3 + 7x_5 = 33$

  $x_1 + 3s + 7t = 33$

  $x_1 = 33 - 3s - 7t$

- $x_4 + \frac{1}{2}x_5 = 4$

  $x_4 = 4 - \frac{1}{2}t$

- $x_2 + 2x_3 - 4x_5 = -27/2$

  $x_2 = -27/2 - 2s + 4t$

SOLUTION SET:

$$(x_1, x_2, x_3, x_4, x_5) = \left(33 - 3s - 7t,\ -\frac{27}{2} - 2s + 4t,\ s,\ 4 - \frac{1}{2}t,\ t\right)$$

WHERE $s, t \in \mathbb{R}$

Question 2. *(10 marks)* Given:.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -1 \\ 4 & 2 \\ -2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 4 \\ 3 & 1 & 3 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & -4 \end{bmatrix}$$

Compute the following if possible.
(a) $A^{-1} + DB$
(b) $C^T - BA$
(c) $AD - B^T$
(d) $\mathrm{tr}(BD)$

a) $A^{-1} + BD = \frac{1}{0-1}\begin{bmatrix} 0 & -1 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & -4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 4 & 2 \\ -2 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 6 & -1 \\ 16 & 3 \end{bmatrix} = \underline{\begin{bmatrix} 6 & 0 \\ 17 & 0 \end{bmatrix}}$

b) NOT POSSIBLE SINCE C^T IS 3X3 AND BA IS 3X2, MATRICES MUST BE SAME SIZE TO SUBTRACT

c) $AD - B^T = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 4 & 2 \\ -2 & 0 \end{bmatrix}^T$

$= \begin{bmatrix} 4 & 2 & -13 \\ 1 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 4 & -2 \\ -1 & 2 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 4 & -2 & -11 \\ 2 & -2 & -3 \end{bmatrix}}$

d) $\mathrm{tr}(BD) = \mathrm{tr}\left(\begin{bmatrix} 0 & -1 \\ 4 & 2 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ 1 & 2 & -4 \end{bmatrix}\right)$

$= \mathrm{tr}\left(\begin{bmatrix} -1 & -2 & 4 \\ 6 & 4 & -20 \\ -2 & 0 & 6 \end{bmatrix}\right) = -1 + 4 + 6 = 9$

Question 3. *(8 marks)*
(a) Find A^{-1} if possible given:

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$$

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - R_1} \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_2]{R_1 - 6R_2} \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 7 & -6 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_1 - 6R_3} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_1 \cdot 1/2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \qquad \therefore \ A^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) Use (a) to solve the following system:

$$\begin{aligned} 2x_1 + 6x_2 + 6x_3 &= 4 \\ 2x_1 + 7x_2 + 6x_3 &= 1 \\ 2x_1 + 7x_2 + 7x_3 &= -3 \end{aligned}$$

CAN BE WRITTEN AS $Ax = b$

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} \Rightarrow \quad x = A^{-1}b$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -3 \\ -4 \end{bmatrix}$$

$$\therefore \ (x_1, x_2, x_3) = (23, -3, -4)$$

Question 4. Given:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -2 & 3 & 1 \end{bmatrix}$$

(a) *(1 mark)* Put A in reduced row echelon form.

$$\begin{bmatrix}0&1&0\\1&0&0\\-2&3&1\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}1&0&0\\0&1&0\\-2&3&1\end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

(b) *(3 marks)* Find elementary matrices $E_1, E_2, \ldots, E_n$ such that $E_n \ldots E_2 E_1 A = I$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}0&1&0\\1&0&0\\0&1&0\end{bmatrix} = E_1 \qquad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3-3R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix} = E_3$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_2 \qquad \therefore\ E_3 E_2 E_1 A = I$$

(c) *(1 mark)* Write A^{-1} as a product of elementary matrices.

$$A^{-1} = E_3 E_2 E_1 = \begin{bmatrix}1&0&0\\0&1&0\\0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$$

(d) *(3 marks)* Write A as a product of elementary matrices. Find these matrices.

$$E_3 E_2 E_1 A = I \Rightarrow E_2 E_1 A = E_3^{-1} \Rightarrow E_1 A = E_2^{-1} E_3^{-1}$$

$$\Rightarrow A = E_1^{-1} E_2^{-1} E_3^{-1}$$

USE INVERSE OPERATIONS TO FIND $E_1^{-1}, E_2^{-1}, E_3^{-1}$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} = E_1^{-1} \qquad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3+3R_2} \begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix} = E_3^{-1}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix} = E_2^{-1} \qquad \therefore\ A = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}$$

Question 5. *(3 marks)* Find A^{-1} if $A^3 - 2A + I = 0$.

$$A^3 - 2A + I = 0$$
$$A^{-1}(A^3 - 2A + I) = A^{-1}0$$
$$A^{-1}A^3 - 2A^{-1}A + A^{-1}I = 0$$
$$A^2 - 2I + A^{-1} = 0$$
$$\therefore A^{-1} = 2I - A^2$$

Question 6. *(3 marks)* Show that $(A-I)^{-1} = I + A + A^2$ if $A^3 = 2I$.

$$(A-I)(I+A+A^2)$$
$$= A(I+A+A^2) - I(I+A+A^2)$$
$$= A + A^2 + A^3 - I - A - A^2$$
$$= A^3 - I$$
$$= 2I - I$$
$$= I$$

$$(I+A+A^2)(A-I)$$
$$= (I+A+A^2)A - (I+A+A^2)I$$
$$= A + A^2 + A^3 - I - A - A^2$$
$$= A^3 - I = 2I - I$$
$$= I$$
$$\therefore (A-I)^{-1} = I + A + A^2$$

Question 7. *(5 marks)* Find numbers a, b, c such that the following matrix is symmetric:

$$A = \begin{bmatrix} 1 & 2c-1 & -2 \\ 2a+c & -2 & c+2b \\ a-c & 1 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2a+c & a-c \\ 2c-1 & -2 & 1 \\ -2 & c+2b & 0 \end{bmatrix}$$

A IS SYMMETRIC WHEN $A^T = A$

$$\therefore \begin{aligned} 2c-1 &= 2a+c \\ -2 &= a-c \\ 1 &= c+2b \end{aligned} \Rightarrow \begin{aligned} 2a \quad\;\; -c &= -1 \\ a \quad\;\; -c &= -2 \\ 2b + c &= 1 \end{aligned}$$

$$\begin{bmatrix} 2 & 0 & -1 & -1 \\ 1 & 0 & -1 & -2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 - 2R_2} \begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & 0 & -1 & -2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \cdot 1/2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore a = 1, \quad b = -1, \quad c = 3$$

Question 8. *(7 marks)* Consider the following system

$$\begin{aligned} x_1 \quad\quad +(a+2)x_3 &= 0 \\ 2x_2 \quad\quad &= -2 \\ (a-2)x_1 + 2x_2 \quad\quad &= a \end{aligned}$$

where a is a real number . If possible find value(s) of a so that the system has
(a) a unique solution
(b) infinitely many solutions
(c) no solutions

AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 1 & 0 & a+2 & 0 \\ 0 & 2 & 0 & -2 \\ a-2 & 2 & 0 & a \end{array}\right] \xrightarrow{R_3-(a-2)R_1} \left[\begin{array}{ccc|c} 1 & 0 & a+2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 2 & -(a+2)(a-2) & a \end{array}\right] \xrightarrow{R_3-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & a+2 & 0 \\ 0 & 2 & 0 & -2 \\ 0 & 0 & -(a+2)(a-2) & a+2 \end{array}\right] \xrightarrow{R_2 \cdot 1/2} \left[\begin{array}{ccc|c} 1 & 0 & a+2 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -(a+2)(a-2) & a+2 \end{array}\right]$$

a) UNIQUE SOLUTION IF $-(a+2)(a-2) \neq 0 \Rightarrow a \neq \pm 2$

b) INFINITELY MANY SOLUTIONS IF $-(a+2)(a-2) = 0$ AND $a+2=0$
$\Rightarrow a = -2$

c) NO SOLUTION IF $-(a+2)(a-2) = 0$ AND $a+2 \neq 0$
$\Rightarrow a = \pm 2$ AND $a \neq -2$
$\Rightarrow a = 2$

Bonus. *(3 marks)* A matrix A is called skew-symmetric if $A^T = -A$. Determine whether or not the matrix $A = [a_{ij}]_{n \times n}$ whose ij-entry is $a_{ij} = i^2 - 2i + 2j - j^2$ is skew symmetric.

LET $B = A^T = [a_{ij}]^T \Rightarrow b_{ij} = a_{ji}$

AND $b_{ij} = a_{ji} = j^2 - 2j + 2i - i^2 = -(i^2 - 2i + 2j - j^2) = -a_{ij}$

$\therefore B = [b_{ij}] = -[a_{ij}] = -A$