Dawson College: **Linear Algebra**: 201-NYC-05 06

April 7, 2010

Last Name: SOLUTIONS

First Name:

Student ID:

Test 2

Question 1. *(6 marks)* Find the determinant of A using cofactor expansion:

$$A = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 4 & 0 & 0 & 2 \\ 0 & 3 & 5 & -3 \\ -1 & 0 & -2 & 4 \end{bmatrix}$$

(COLUMN 2)

$$\det A = -0 + 0 - 3 \det \begin{bmatrix} 3 & 1 & 2 \\ 4 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix} + 0$$

(COLUMN 2)

$$= -3\left((-1) \det \begin{bmatrix} 4 & 2 \\ -1 & 4 \end{bmatrix} - (-2) \det \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} \right)$$

$$= -3 \left(-(16+2) + 2(6-8) \right)$$

$$= -3(-18-4)$$

$$= 66$$

Question 2. *(5 marks)*
Given that the following system has a unique solution solve the system using **Cramer's rule**:

$$\begin{aligned} a_1x &+ b_1y &+ c_1z &= b_1 \\ a_2x &+ b_2y &+ c_2z &= b_2 \\ a_3x &+ b_3y &+ c_3z &= b_3 \end{aligned}$$

$$x = \frac{\det\begin{bmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{bmatrix}}{\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = \frac{0}{\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = 0$$

NOT ZERO SINCE THERE IS A UNIQUE SOLUTION

$$y = \frac{\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}}{\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = 1$$

$$z = \frac{\det\begin{bmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{bmatrix}}{\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = \frac{0}{\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}} = 0$$

$\therefore \quad (x, y, z) = (0, 1, 0)$

Question 3. *(6 marks)* Find the determinant of A by first putting the matrix in row echelon from:

$$A = \begin{bmatrix} 0 & -2 & 6 \\ 3 & 6 & 4 \\ 3 & 9 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 3 & 6 & 4 \\ 0 & -2 & 6 \\ 3 & 9 & -1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 3 & 6 & 4 \\ 0 & -2 & 6 \\ 0 & 3 & -5 \end{bmatrix} \xrightarrow[R_2 \cdot (-1/2)]{R_1 \cdot 1/3}$$

$$\begin{bmatrix} 1 & 2 & 4/3 \\ 0 & 1 & -3 \\ 0 & 3 & -5 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \begin{bmatrix} 1 & 2 & 4/3 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{R_3 \cdot (1/4)} \begin{bmatrix} 1 & 2 & 4/3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = B$$

$\therefore \ (-1)(1/3)(-1/2)(1/4) \det A = \det B = 1$

$\therefore \ \det A = 3 \cdot 2 \cdot 4 = 24$

Question 4. *(8 marks)* Given that A and B are 4×4 matrices with $\det(2A) = 32$ and $\det(B) = -3$ find:

(a) $\det(A^2 B^T)$

(b) $\det((3A)^{-1}(3B)^3)$

(c) $\det((\mathrm{adj}(B))^{-1}(2AB)^T)$

$\det(2A) = 2^4 \det A = 32$

$\therefore \ \det A = 2$

a) $\det(A^2 B^T) = \det(A^2)\det(B^T) = (\det A)^2 \det(B) = (2)^2(-3)$

$= -12$

b) $\det((3A)^{-1}(3B)^3) = \det((3A)^{-1})(\det(3B))^3 = \dfrac{1}{\det(3A)} \cdot (3^4 \det B)^3$

$= \dfrac{1}{3^4 \det(A)} \cdot 3^{12}(\det B)^3 = \dfrac{1}{(2)} \cdot 3^8(-3)^3$

$= -\dfrac{177147}{2}$

c) $\det((\mathrm{adj}\, B)^{-1}(2AB)^T) = \det((\mathrm{adj}\, B)^{-1}) \cdot \det((2AB)^T)$

$= \dfrac{1}{\det(\mathrm{adj}\, B)} \cdot \det(2AB) = \dfrac{1}{\det(B)^{4-1}} \cdot 2^4 \det A \det B$

$= \dfrac{1}{(-3)^3} \cdot 2^4(2)(-3) = \dfrac{32}{9}$

Question 5. *(6 marks)* If the entries in each row of a 3×3 matrix A add up to zero, show that $\det(A) = 0$. (Hint: consider the product AX where X is the 3×1 matrix whose entries are all 1.)

LET $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ THEN $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}+a_{12}+a_{13} \\ a_{21}+a_{22}+a_{23} \\ a_{31}+a_{32}+a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

IF $\det(A) \neq 0$ THEN A IS INVERTABLE SO

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A^{-1}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ WHICH IS NOT POSSIBLE

$\therefore$ A IS NOT INVERTABLE

$\therefore$ $\det(A) = 0$

Question 6. *(6 marks)* Given $\vec{u} = (3, -2, 0)$, $\vec{v} = (1, 1, 2)$ $\vec{w} = (2, -1, 3)$, find:
(a) $||3\vec{u} - 5\vec{v}||$
(b) a unit vector in the opposite direction of $\vec{w}$
(c) a vector perpendicular to both $\vec{u}$ and $\vec{w}$.

a) $3\vec{u} - 5\vec{v} = (9, -6, 0) - (5, 5, 10) = (4, -11, -10)$

$\therefore \; ||3\vec{u} - 5\vec{v}|| = \sqrt{(4)^2 + (-11)^2 + (10)^2} = \sqrt{237}$

b) $-\vec{w} = (-2, 1, -3)$, $\frac{1}{||-\vec{w}||}(-\vec{w}) = \frac{1}{\sqrt{4+1+9}}(-2, 1, -3) = \left(\frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

c) $\vec{u} \times \vec{w} = \left(\begin{vmatrix} -2 & -1 \\ 0 & 3 \end{vmatrix}, -\begin{vmatrix} 3 & 2 \\ 0 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} \right) = (-6, -9, 1)$

$\begin{matrix} 3 & 2 \\ -2 & -1 \\ 0 & 3 \end{matrix}$

Question 7. *(9 marks)* Determine whether the following statements are true or false in general. Prove the statement if it is true otherwise find an example to show that it is false.
(a) $\det(A+B) = \det(A) + \det(B)$
(b) $\vec{\mathbf{v}} \cdot (\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = 0$
(c) parallel vectors with the same length are equal
(d) $\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{v}} \times \vec{\mathbf{u}}$

a) LET $A = B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ THEN $\det(A+B) = \det \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$= 2^3 = 8$

BUT $\det(A) + \det(B) = 1 + 1 = 2$

FALSE.

b) LET $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$

THEN $\vec{v} \cdot (\vec{u} \times \vec{v}) = \det \begin{bmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = 0$ SINCE $R_1 = R_3$

TRUE

c) FALSE $\vec{u} = (1, 0, 0)$ AND $\vec{v} = (-1, 0, 0)$ ARE PARALLEL AND $\|\vec{u}\| = \|\vec{v}\| = 1$ BUT $\vec{u} \neq \vec{v}$

d) LET $\vec{u} = (1, 0, 0)$ AND $\vec{v} = (0, 1, 0)$

THEN $\vec{u} \times \vec{v} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (0, 0, 1)$

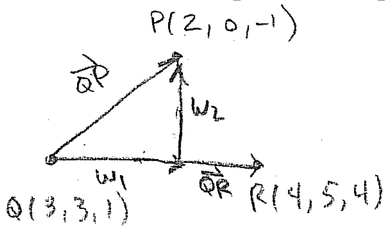
AND $\vec{v} \times \vec{u} = \left(\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, -\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \right) = (0, 0, -1)$

SO $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$

FALSE

(IN FACT $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$)

Question 8. *(6 marks)* Use projections to find the distance between the point P$(2,0,-1)$ and the line that passes throught the points $Q(3,3,1)$ and $R(4,5,4)$.



$\vec{u} = \overrightarrow{QP} = (-1,-3,-2)$, $\vec{v} = \overrightarrow{QR} = (1,2,3)$

$\vec{w}_1 = \text{proj}_{\vec{v}}\, \vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\,\vec{v} = \dfrac{-13}{14}(1,2,3)$

$= \left(\dfrac{-13}{14}, \dfrac{-26}{14}, \dfrac{-39}{14}\right)$

$\vec{W}_2 = \vec{u} - \vec{w}_1 = (-1,-3,-2) - \left(\dfrac{-13}{14}, \dfrac{-26}{14}, \dfrac{-39}{14}\right) = \left(\dfrac{-1}{14}, \dfrac{-16}{14}, \dfrac{11}{14}\right)$

$\therefore$ DISTANCE $= \|\vec{w}_2\| = \sqrt{\left(\frac{-1}{14}\right)^2 + \left(\frac{-16}{14}\right)^2 + \left(\frac{11}{14}\right)^2}$

$= \dfrac{1}{14}\sqrt{1 + 256 + 121} = \dfrac{1}{14}\sqrt{378} = \dfrac{3}{14}\sqrt{42}$

Question 9. *(4 marks)* Show that if A is invertable and AB is not invertable then B is not invertable.

A IS INVERTABLE $\Rightarrow$ $\det A \neq 0$

$\because$ AB IS NOT INVERTABLE $\Rightarrow 0 = \det(AB) = \det A \det B$

BUT $\det A \neq 0$ $\Rightarrow$ $\det B = 0$

$\therefore$ B IS NOT INVERTABLE.

Bonus. *(4 marks)*
(i) Show that adding a multiple of the last row (say k times the last row) to the second last row does not affect the determinant of a 2×2 matrix.
(ii). Suppose that adding a multiple of the last row to the second last row does not affect the determinant of an $n\times n$ matrix. Show that this means that adding a multiple of the last row to the second last row does not affect the determinant of an $(n+1)\times(n+1)$ matrix.

You have, in fact, proved that adding a multiple of the last row to the second last row does not affect the determinant for any sized matrix. Why is this the case? This type of proof is called a proof by induction.

i) LET $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ THEN $\det\begin{bmatrix} a_1+kb_1 & a_2+kb_2 \\ b_1 & b_2 \end{bmatrix} =$

$= a_1b_2 + kb_1b_2 - b_1a_2 - kb_1b_2 = a_1b_2 - b_1a_2 = \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

AS REQUIRED.

ii) LET $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1(n+1)} \\ a_{21} & a_{22} & \cdots & a_{2(n+1)} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n(n+1)} \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{bmatrix}$

THEN $\det\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1\,n+1} \\ a_{21} & a_{22} & \cdots & a_{2\,(n+1} \\ \vdots & & & \\ b_1 & b_2 & \cdots & b_{n+1} \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{bmatrix} =$

WHERE $b_i = a_{ni} + k a_{(n+1)i}$

(IE: $R_n + kR_{n+1}$)

$$= a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2\,n+1} \\ \vdots & & \\ b_1 & \cdots & b_{n+1} \\ a_{n+1\,1} & \cdots & a_{n+1\,n+1} \end{vmatrix} - a_{21} \begin{vmatrix} a_{21} & a_{31} & \cdots & a_{2n+1} \\ \vdots & & & \\ b_1 & b_3 & \cdots & b_{n+1} \\ a_{n+1\,1} & a_{n+1\,2} & \cdots & a_{n+1\,n+1} \end{vmatrix} + \cdots$$

$$+ a_{1\,n+1} \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ b_1 & \cdots & b_n \\ a_{n+1\,1} & \cdots & a_{n+1\,n} \end{vmatrix}$$

(FOR EACH DETERMINANT $R_n = R_n + k R_{n+1}$ SO BY OUR ASSUMPTION)

$$= a_{11} \begin{vmatrix} a_{22} & \cdots & a_{2(n+1)} \\ \vdots & & \\ a_{n1} & \cdots & a_{n\,n+1} \\ a_{n+1\,1} & \cdots & a_{n+1\,n+1} \end{vmatrix} + \cdots + a_{1\,n+1} \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & & a_{nn} \\ a_{n+1\,1} & & a_{n+1\,n} \end{vmatrix}$$

$= \det A$ AS REQUIRED.

WE HAVE SHOW THAT THE STATEMENT IS TRUE FOR A 2X2. BUT ii) MEANS THAT IT IS TRUE FOR A 3X3. BUT AGAIN ii) MEANS THAT IT IS TRUE FOR A 4X4 ETC. SO, THE STATEMENT IS TRUE FOR ANY SIZED MATRIX.