Dawson College: **Linear Algebra**: 201-NYC-05 06

May 5, 2010

Last Name: SOLUTIONS

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Test 3

Question 1. *(2 marks)* Determine whether the following line $(x,y,z) = (3+(3/7)t,\ 4-(2/14)t,\ (8/7)t)$ and the plane $-(2/3)x+(2/9)y-(16/9)z+12=0$ and are perpendicular, parallel or neither.

$$\vec{v} = \left(\frac{3}{7}, -\frac{2}{14}, \frac{8}{7}\right) \qquad \vec{n} = \left(-\frac{2}{3}, \frac{2}{9}, -\frac{16}{9}\right)$$

$$\vec{v} = k\vec{n}\ ?$$

$$\left(\frac{3}{7}, -\frac{2}{14}, \frac{8}{7}\right) = \left(-\frac{2}{3}k, \frac{2}{9}k, \frac{-16}{9}k\right)$$

$$\Rightarrow \quad \frac{3}{7} = -\frac{2}{3}k \Rightarrow -\frac{19}{14} = k$$

$$-\frac{2}{14} = \frac{2}{9}k \Rightarrow -\frac{9}{14} = k$$

$$\frac{8}{7} = -\frac{16}{9}k \Rightarrow -\frac{19}{14} = k$$

$\therefore$ THE LINE AND THE PLANE ARE PERPENDICULAR.

Question 2. *(7 marks)* Given the point P$(1,-3,2)$ and the plane $3x+4y-2z+1=0$
(a) Find the equations of the line through P that is perpendicular to the plane.
(b) Find the intersection of the line found in **(a)** and the plane.
(c) Use part **(b)** to find the distance between P and the plane.

a) $\vec{v} = \vec{n} = (3,4,-2) \quad \therefore (x,y,z) = (1+3t, -3+4t, 2-2t) \; t\in\mathbb{R}$

b) $3(1+3t)+4(-3+4t)-2(2-2t)+1=0$

$3+9t-12+16t-4+4t+1=0$

$-12+29t=0 \Rightarrow t = 12/29$

$\therefore Q(x,y,z) = \left(1+3\left(\frac{12}{29}\right), -3+4\left(\frac{12}{29}\right), 2-2\left(\frac{12}{29}\right)\right)$

$= \left(\frac{65}{29}, \frac{-39}{29}, \frac{34}{29}\right)$

c) DISTANCE $= \|\vec{PQ}\| = \left\|\left(\frac{65}{29}-1, \frac{-39}{29}-(-3), \frac{34}{29}-2\right)\right\|$

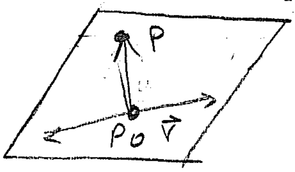
$= \left\|\left(\frac{36}{29}, \frac{48}{29}, -\frac{24}{29}\right)\right\|$

$= \left|\frac{1}{29}\right| \|(36,48,-24)\|$

$= \frac{1}{29}\sqrt{36^2+48^2+(-24)^2}$

$= \frac{1}{29}\sqrt{4176} = \frac{12}{29}\sqrt{29}$ units

Question 3. *(5 marks)* Find the equation of the plane containing the line $(x,y,z) = (5-t,\ 2-4t,\ -1+2t)$ and the point $\mathrm{P}(1,2,1)$.



$P_0(5,2,-1)$ IS A POINT ON THE LINE AND PLA

$\therefore\ \overrightarrow{PP_0} = (4, 0, -2)$, $\vec{v} = (-1,-4,2)$

DIRECTION VECTOR OF LINE

$$\vec{n} = \overrightarrow{PP_0} \times \vec{v} = \left(\begin{vmatrix} 0 & -4 \\ -2 & 2 \end{vmatrix}, -\begin{vmatrix} 4 & -1 \\ -2 & 2 \end{vmatrix}, \begin{vmatrix} 4 & -1 \\ 0 & -4 \end{vmatrix} \right)$$

$$= (-8, -6, -16)$$

$$\therefore\ -8x - 6y - 16z + d = 0$$

$$-8(1) - 6(2) - 16(1) + d = 0$$

$$\therefore\ d = 36$$

EQUATION OF PLANE: $-8x - 6y - 16z + 36 = 0$

Question 4. *(5 marks)* Show that the points $(1,1,-5)$, $(2,1,-8)$, $(1,3,-1)$, and $(0,4,4)$ lie on the same plane.

P_1 P_2 P_3 P_4

$\overrightarrow{P_1P_2} = (1, 0, -3)$, $\overrightarrow{P_1P_3} = (0, 2, 4)$

$$\therefore\ \vec{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \left(\begin{vmatrix} 0 & 2 \\ -3 & 4 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ -3 & 4 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \right)$$

$\begin{matrix} 1 \\ 0 \\ -3 \end{matrix}$ $\begin{matrix} 0 \\ 2 \\ 4 \end{matrix}$

$$= (6, -4, 2)$$

$$\therefore\ 6x - 4y + 2z + d = 0$$

$6(1) - 4(1) + 2(-5) + d = 0$ (USING P_1)

$$\therefore\ d = 8$$

$6x - 4y + 2z + 8 = 0$ ← PLANE CONTAINING P_1, P_2, P_3

IS P_4 ON THE PLANE?

$6(0) - 4(4) + 2(4) + 8 = -16 + 16 = 0$ YES.

$\therefore$ ALL FOUR POINTS LIE ON THE SAME PLANE.

Question 5. *(7 marks)* Let V be the set of four-tuples of the form $(-1, 0, x, y)$ where x and y are real numbers. Define addition on the set by

$$(-1, 0, x_1, y_1) + (-1, 0, x_2, y_2) = (-1, 0, x_1 + x_2, y_1 + y_2)$$

and define scalar multiplication by

$$k(-1, 0, x, y) = \cancel{(-1, 0, x, y)}. = (-1, 0, kx, ky)$$

Check vector space axioms 2, 4, 5, and 10. If all other axioms are satisfied is V a vector space?

AXIOM 2 LET $\vec{u} = (-1, 0, x_1, y_1)$, $\vec{v} = (-1, 0, x_2, y_2) \in V$

THEN $\vec{u} + \vec{v} = (-1, 0, x_1 + x_2, y_1 + y_2)$

$\vec{v} + \vec{u} = (-1, 0, x_2 + x_1, y_2 + y_1) = (-1, 0, x_1 + x_2, y_1 + y_2)$

$\therefore \vec{u} + \vec{v} = \vec{v} + \vec{u}$

AXIOM 4 LET $\vec{0} = (-1, 0, 0, 0)$ THEN $\vec{0} \in V$ AND

$\vec{0} + \vec{u} = (-1, 0, 0, 0) + (-1, 0, x_1, y_1) = (-1, 0, 0 + x_1, 0 + y_1)$

$= (-1, 0, x_1, y_1) = \vec{u}$

AND $\vec{0} + \vec{u} = (-1, 0, 0, 0) + (-1, 0, x_1, y_1) = (-1, 0, x_1, y_1) = \vec{u}$

$\therefore \vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$

AXIOM 5 IF $\vec{u} = (-1, 0, x_1, y_1)$ LET $-\vec{u} = (-1, 0, -x_1, -y_1)$

THEN $-\vec{u} \in V$ AND

$\vec{u} + (-\vec{u}) = (-1, 0, x_1, y_1) + (-1, 0, -x_1, -y_1)$

$= (-1, 0, x_1 + (-x_1), y_1 + (-y_1)) = (-1, 0, 0, 0) = \vec{0}$

AND $(-\vec{u}) + \vec{u} = (-1, 0, (-x_1) + x_1, (-y_1) + y_1) = (-1, 0, 0, 0) = \vec{0}$

AXIOM 10 $1\vec{u} = 1(-1, 0, x_1, y_1) = (-1, 0, 1x_1, 1y_1)$

$= (-1, 0, x_1, y_1) = \vec{u}$

IF ALL AXIOMS ARE SATISFIED THEN V IS A VECTOR SPACE.

Question 6. *(12 marks)*
(a) Is the set of 2×2 diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

a subspace of M_{22}?

LET $\vec{u} = \begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a_2 & 0 \\ 0 & b_2 \end{bmatrix}$ THEN $\vec{u}+\vec{v} = \begin{bmatrix} a_1+a_2 & 0 \\ 0 & b_1+b_2 \end{bmatrix}$

WHICH IS DIAGONAL.

$\therefore$ CLOSED UNDER ADDITION

$k\vec{u} = k\begin{bmatrix} a_1 & 0 \\ 0 & b_1 \end{bmatrix} = \begin{bmatrix} ka_1 & 0 \\ 0 & kb_1 \end{bmatrix}$ WHICH IS DIAGONAL

$\therefore$ CLOSED UNDER SCALAR MULTIPLICATION.

$\therefore$ IT IS A SUBSPACE.

(b) Is the set of 2×2 matrices

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

where $a+b+c=-1$ a subspace of M_{22}?

LET $\vec{u} = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix}$ $a_1+b_1+c_1=-1$, $\vec{u} = \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix}$, $a_2+b_2+c_2=-$

$\vec{u}+\vec{v} = \begin{bmatrix} a_1 & b_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ 0 & c_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ 0 & c_1+c_2 \end{bmatrix}$

AND $(a_1+a_2)+(b_1+b_2)+(c_1+c_2) =$

$(a_1+b_1+c_1)+(a_2+b_2+c_2) = -1+(-1) = -1$

NOT CLOSED UNDER ADDITION

(TAKE $\vec{u} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ $v = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$)

$\therefore$ NOT A SUBSPACE.

(c) Is the set of $n \times n$ matrices A such that $A^T = A^{-1}$ a subspace of M_{nn}?

LET $A \in M_{nn}$ SUCH THAT $A^T = A^{-1}$

THEN $(kA)^T = kA^T = kA^{-1}$

BUT $(kA^{-1}) = \frac{1}{k} A^{-1}$

SO $(kA)^T \neq (kA)^{-1}$ NECESSARILY

COUNTER EXAMPLE: LET $A = I$ THEN $A^T = I^T = I$
$= I^{-1}$

LET $k = 5$. THEN $(5A)^T = (5I)^T = 5I$

BUT $(5A)^{-1} = (5I)^{-1} = \frac{1}{5} I$

SO $(5A)^T \neq (5A)^{-1}$

$\therefore$ NOT CLOSED UNDER SCALAR MULTIPLICATION.

$\therefore$ NOT A SUBSPACE.

Question 7. *(5 marks)* Do the vectors $\vec{\mathbf{u}} = (3,-1,1)$, $\vec{\mathbf{v}} = (2,5,0)$, $\vec{\mathbf{w}} = (3,2,-2)$ span $\mathbb{R}^3$?

LET (x, y, z) BE AN ELEMENT OF $\mathbb{R}^3$

CAN WE FIND k_1, k_2, k_3 SUCH THAT

$$(x, y, z) = k_1\vec{u} + k_2\vec{v}, + k_3\vec{w}$$
$$= k_1(3,-1,1) + k_2(2,5,0) + k_3(3,2,-2)$$

$$\Leftrightarrow \quad \begin{aligned} x &= 3k_1 + 2k_2 + 3k_3 \\ y &= -k_1 + 5k_2 + 2k_3 \\ z &= k_1 \qquad\quad - 2k_3 \end{aligned}$$

$$\Leftrightarrow \quad \begin{bmatrix} 3 & 2 & 3 \\ -1 & 5 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

LET $A = \begin{bmatrix} 3 & 2 & 3 \\ -1 & 5 & 2 \\ 1 & 0 & -2 \end{bmatrix}$ THIS SYSTEM HAS A SOLUTION (IE WE CAN FIND k_1, k_2, k_3) FOR ANY $x, y, z \in \mathbb{R}$ IF AND ONLY IF $\det(A) \neq 0$

$$\det A = \det\begin{bmatrix} 3 & 2 & 3 \\ -1 & 5 & 2 \\ 1 & 6 & -2 \end{bmatrix} = \det\begin{bmatrix} 2 & 3 \\ 5 & 2 \end{bmatrix} - 2\det\begin{bmatrix} 3 & 2 \\ -1 & 5 \end{bmatrix}$$

$$= (4-15) - 2(15+2) = -45 \neq 0$$

$\therefore$ $\vec{u}, \vec{v}$ AND $\vec{w}$ SPAN $\mathbb{R}^3$

Question 8. *(5 marks)* Let $p_1 = 1 + 2x - 4x^2$, $p_2 = 2 + x - 3x^2$, and $p_3 = -2 + 2x + x^2$. Is $p =$ $9x+33x^2$ in span$\{p_1, p_2, p_3\}$? (Note: The set of all quadratic polynomials $a + bx + cx^2$ with usual addition and scalar multiplication is a vector space.)

$$9x + 33x^2 = k_1 p_1 + k_2 p_2 + k_3 p_3$$

$$= k_1(1 + 2x - 4x^2) + k_2(2 + x - 3x^2) + k_3(-2 + 2x + x^2)$$

$$= (k_1 + 2k_2 - 2k_3) + (2k_1 + k_2 + 2k_3)x + (-4k_1 - 3k_2 + k_3)x^2$$

$\Rightarrow$

$$k_1 + 2k_2 - 2k_3 = 0$$
$$2k_1 + k_2 + 2k_3 = 9$$
$$-4k_1 - 3k_2 + k_3 = 33$$

$$\therefore \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 1 & 2 & 9 \\ -4 & -3 & 1 & 33 \end{bmatrix} \xrightarrow[R_3 + 4R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -3 & 6 & 9 \\ 0 & 5 & -7 & 33 \end{bmatrix} \xrightarrow{R_2 \cdot (-1/3)}$$

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & -2 & -3 \\ 0 & 5 & -7 & 33 \end{bmatrix} \xrightarrow[R_3 - 5R_2]{R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 3 & 48 \end{bmatrix} \xrightarrow{R_3 \cdot (1/3)}$$

$$\begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 16 \end{bmatrix} \xrightarrow[]{\substack{R_1 - 3R_3 \\ R_2 + 2R_3}} \begin{bmatrix} 1 & 0 & 0 & -26 \\ 0 & 1 & 0 & 29 \\ 0 & 0 & 1 & 16 \end{bmatrix}$$

$$\therefore k_1 = -26, \quad k_2 = 29, \quad k_3 = 16$$

YES p IS IN span$\{p_1, p_2, p_3\}$ SINCE

$$p = -26p_1 + 29p_2 + 16p_3$$

Bonus. *(3 marks)* The union of two sets, W_1 and W_2, is defined as $W_1 \cup W_2 = \{x \mid x \in W_1 \text{ or } x \in W_2\}$. If W_1 and W_2 are subspaces of a vector space V is $W_1 \cup W_2$ always a subspace of V? Prove that it is or give a counter example to show that it isn't.

LET $W_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

$W_2 = \left\{ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$ BOTH SUBSPACES OF M_{22}.

NOW $\vec{V}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in W_1$ AND $\vec{V}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in W_2$

AND SO $\vec{V}_1$ AND $\vec{V}_2$ ARE IN $W_1 \cup W_2$

BUT $\vec{V}_1 + \vec{V}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ WHICH IS NOT IN W_1 OR W_2

$\therefore$ $\vec{V}_1 + \vec{V}_2$ IS NOT IN $W_1 \cup W_2$

SO $W_1 \cup W_2$ IS NOT CLOSED UNDER ADDITION

$\therefore$ $W_1 \cup W_2$ IS NOT A SUBSPACE OF M_{22}.

NO, $W_1 \cup W_2$ IS NOT ALWAYS A SUBSPACE OF V.

Vector Space Axioms

1- If u and v are objects in V, then $u+v$ is in V.

2- $u+v=v+u$

3- $u+(v+w)=(u+v)+w$

4- There is an object 0_v called a zero object for V such that $0_v+u=u$ for all u in V.

5- For each u in V, there is an object $-u$ in V called a negative of u such that $u+(-u)=0_v$

6- If k is any scalar and u is any object in V, then ku is in V

7- $k(u+v) = ku + kv$

8- $(k+m)u = ku + mu$

9- $k(mu) = (km)u$

10- $1u = u$