Dawson College: Calculus II: 201-NYB-05-S02

Name: Y. Lamontagne
Student ID:

Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Formula:

$\sum_{i=1}^{n} c = cn$ where c is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Question 1. *(5 marks)* Evaluate using the definition of the definite integral.

$$\int_1^2 -3x^2+2x-1\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(1+\tfrac{i}{n})\Delta x$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(1+\tfrac{i}{n})$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[-3(1+\tfrac{i}{n})^2 + 2(1+\tfrac{i}{n}) - 1\right]$$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$x_i = a + i\Delta x$$

$$x_i = 1 + \frac{i}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[-3\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) + 2 + \frac{2i}{n} - 1\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[-\frac{3i^2}{n^2} - \frac{6i}{n} - 3 + 2 - 1 + \frac{2i}{n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\frac{-3}{n^2}\sum_{i=1}^{n} i^2 - \frac{4}{n}\sum_{i=1}^{n} i - \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\frac{-\cancel{3}}{n^2} \frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}\,2} - \frac{\cancel{4}^{\,2}}{n} \frac{\cancel{n}(n+1)}{\cancel{2}\,1} - 2\cancel{n}\right]$$

$$= \lim_{n\to\infty} -\frac{(n+1)}{n} \cdot \frac{(2n+1)}{2n} - \frac{2(n+1)}{n} - 2$$

$$= -1 - 2 - 2$$

$$= -5$$

Question 2. *(5 marks)* Evaluate the following definite integral:

$$\int_1^2 \frac{(x+1)(x-2)}{\sqrt{x}} - e^x\, dx$$

$$= \int_1^2 \frac{x^2-x-2}{\sqrt{x}} - e^x dx$$

$$= \int_1^2 (x^2-x-2)x^{-\frac{1}{2}} - e^x dx$$

$$= \int_1^2 x^{3/2} - x^{1/2} - 2x^{-\frac{1}{2}} - e^x dx$$

$$= \left[\frac{2x^{5/2}}{5} - \frac{2x^{3/2}}{3} - 4x^{1/2} - e^x \right]_1^2$$

$$= \frac{2 2^{5/2}}{5} - \frac{2 2^{3/2}}{3} - 4 2^{1/2} - e^2 - \frac{2}{5} + \frac{2}{3} + 4 + e$$

$$= \frac{2\sqrt{32}}{5} - \frac{2\sqrt{8}}{3} - 4\sqrt{2} - e^2 - \frac{2}{5} + \frac{2}{3} + 4 + e$$

Question 3. *(5 marks)* Evaluate the following indefinite integral:

$$\int z^2\sqrt{z-1}\, dz = \int (u+1)^2 \sqrt{u}\, du$$

$u = z-1$

$du = dz$

$z = u+1$

$$= \int (u^2+2u+1)\sqrt{u}\, du$$

$$= \int u^{5/2} + 2u^{3/2} + u^{1/2} du$$

$$= \frac{2u^{7/2}}{7} + \frac{4u^{5/2}}{5} + \frac{2u^{3/2}}{3} + C$$

$$= \frac{2(z-1)^{7/2}}{7} + \frac{4(z-1)^{5/2}}{5} + \frac{2(z-1)^{3/2}}{3} + C.$$

Question 4. *(5 marks)* Find the average value of the function

$f(x) = \frac{x}{x^4+9}$

on the interval $[0, \sqrt{3}]$

average value of function $= \frac{1}{b-a}\int_a^b f(x)dx$

$= \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \frac{x}{(x^2)^2+9}dx$

$= \frac{1}{\sqrt{3}}\int_0^3 \frac{\cancel{x}}{u^2+9}\frac{du}{2\cancel{x}} = \frac{1}{\sqrt{3}}\int_0^3 \frac{\cancel{x}}{u^2+9}\frac{du}{2\cancel{x}}$

$u = x^2$

$u(0) = 0^2 = 0$

$u(\sqrt{3}) = (\sqrt{3})^2 = 3$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$= \frac{1}{2\sqrt{3}}\int_0^3 \frac{1}{u^2+9}du$

$= \frac{1}{2\sqrt{3}}\left[\frac{1}{3}\arctan\frac{u}{3}\right]_0^3$

$= \frac{1}{2\sqrt{3}}\left[\frac{1}{3}\arctan\frac{3}{3} - \frac{1}{3}\arctan\frac{0}{3}\right] = \frac{1}{2\sqrt{3}}\frac{1}{3}\frac{\pi}{4}$

$= \frac{\pi}{24\sqrt{3}}$

Question 5. *(5 marks + 1 bonus mark to simplify completely)* Evaluate the following expression:

$$\frac{d}{dx}\left[\int_{-\sin x}^{\sin x} t\cos t^9 dt\right]$$

method 1: notice $f(x) = x\cos x^9$ is an odd function since

$f(-x) = -x\cos(-x)^9 = -x\cos(-x^9) = -x\cos x^9 = -f(x)$.

$\therefore \frac{d}{dx}\left[\int_{-\sin x}^{\sin x} t\cos t^9 dt\right] = \frac{d}{dx}[0] = 0$

method 2: $g(x) = \int_{-\sin x}^{\sin x} t\cos t^9 dt = \int_{-\sin x}^{0} t\cos t^9 dt + \int_0^{\sin x} t\cos t^9 dt$

where $f(x) = \int_0^x t\cos t^9 dt$ $\quad = -\int_0^{-\sin x} t\cos t^9 + \int_0^{\sin x} t\cos t^9 dt$

$f'(x) = x\cos x^9$ by 2^{nd} FTC $\quad = -f(g_1(x)) + f(g_2(x))$

$g_1(x) = -\sin x \quad g_1'(x) = -\cos x$

$g_2(x) = \sin x \quad g_2'(x) = \cos x$

$\therefore g'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$

$= -(-\sin x)\cos(-\sin x)^9(-\cos x)$

$+ \sin x\cos(\sin x)^9\cos x$

$= -\sin x\cos(\sin x)^9\cos x + \sin x\cos(\sin x)^9\cos x$

$= 0$ ↑ cos is even

Question 6. *(5 marks)* Suppose $f(x)$ is continous over the real numbers and

$$\int_0^1 f(x)\,dx = 131.$$

Evaluate

$$\int_0^{\pi/6} \cos(3x)\, f(\sin(3x))\, dx. \quad = \int_0^1 f(u)\,\frac{du}{3} = \frac{1}{3}\int_0^1 f(u)\,du$$

$$= \frac{1}{3}(131)$$

$u = \sin(3x)$

$du = \cos(3x)\cdot 3\,dx$

$\frac{du}{3} = \cos(3x)\,dx$

$u(0) = (3(0)) = 0$

$u(\frac{\pi}{6}) = \sin(\frac{3\pi}{6}) = 1$

Question 7. *(5 marks)* Evaluate the following definite integral.

$$\int x^2 \arcsin x\, dx \qquad = uv - \int v\,du$$

$u = \arcsin x \qquad du = \frac{1}{\sqrt{1-x^2}}\,dx$

$v = \frac{x^3}{3} \qquad dv = x^2\,dx$

$u = 1 - x^2 \iff x^2 = 1 - u$

$du = -2x\,dx$

$\frac{du}{-2x} = dx$

$$= \frac{x^3}{3}\arcsin x - \int \frac{x^3}{3}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^3 \arcsin x}{3} - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^3 \arcsin x}{3} - \frac{1}{3}\int \frac{x^{\cancel{3}\,2}}{\sqrt{u}}\,\frac{du}{-2\cancel{x}}$$

$$= \frac{x^3 \arcsin x}{3} + \frac{1}{6}\int \frac{1-u}{\sqrt{u}}\,du$$

$$= \frac{x^3 \arcsin x}{3} + \frac{1}{6}\int (1-u)\,u^{-\frac{1}{2}}\,du$$

$$= \frac{x^3 \arcsin x}{3} + \frac{1}{6}\int u^{-1/2} - u^{1/2}\,du$$

$$= \frac{x^3 \arcsin x}{3} + \frac{1}{\cancel{6}3}\left[\frac{\cancel{2}u^{\frac{1}{2}}}{1} - \frac{\cancel{2}u^{3/2}}{3}\right] + C$$

$$= \frac{x^3 \arcsin x}{3} + \frac{(1-x^2)^{1/2}}{3} - \frac{(1-x^2)^{3/2}}{9} + C$$

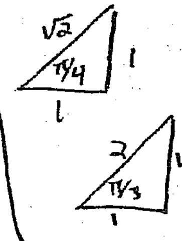
Question 8. *(5 marks)* Evaluate the following definite integral.

$$\int_{\pi/6}^{\pi/8} \theta \sec 2\theta \tan 2\theta \, d\theta \qquad = \left[uv\right]_{\pi/6}^{\pi/8} - \int_{\pi/6}^{\pi/8} v \, du$$

$u = \theta \qquad du = d\theta$

$V = \frac{\sec 2\theta}{2} \qquad dv = \sec 2\theta \tan 2\theta \, d\theta$

$$= \left[\frac{\theta \sec 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{8}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{8}} \frac{\sec 2\theta}{2} \, d\theta$$



$$= \left[\frac{\pi/8 \sec 2(\pi/8)}{2}\right] - \left[\frac{\pi/6 \sec 2(\pi/6)}{2}\right] - \frac{1}{4}\left[\ln|\sec 2\theta + \tan 2\theta|\right]_{\pi/6}^{\pi/8}$$

$$= \frac{\pi \sec \pi/4}{16} - \frac{\pi \sec \frac{\pi}{3}}{12} - \frac{1}{4} \ln|\sec \tfrac{\pi}{4} + \tan \tfrac{\pi}{4}| + \frac{1}{4} \ln|\sec \tfrac{\pi}{3} + \tan \tfrac{\pi}{3}|$$

$$= \frac{\pi\sqrt{2}}{16} - \frac{\pi \, 2}{12} - \frac{1}{4}\ln|\sqrt{2}+1| + \frac{1}{4}\ln|2+\sqrt{3}| \qquad = \frac{\pi\sqrt{2}}{16} - \frac{\pi}{6} + \ln \sqrt[4]{\frac{2+\sqrt{3}}{\sqrt{2}+1}}$$

Question 9. *(5 marks)* Prove: If $f(x)$ is a continous function then

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

$$\int_a^b f(x)+g(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i) + g(x_i)]\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i)\Delta x + g(x_i)\Delta x]$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\Delta x + \sum_{i=1}^{n} g(x_i)\Delta x\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x$$

$$= \int_a^b f(x)dx + \int_a^b g(x)dx$$

Bonus Question. *(3 marks)*
If $f(x)$ is a continuous function on a certain domain and satisfies

$$0 = \int_{101}^{x} f(t)\,dt - \operatorname{arcsec}(\ln x) - \int_{x}^{102} (t^2+1) f(t)\,dt$$

then find $f(x)$ and state its domain.

$$0 = \int_{101}^{x} f(t)dt - \operatorname{arcsec}(\ln x) + \int_{102}^{x} (t^2+1) f(t) dt$$

$$\frac{d}{dx}[0] = \frac{d}{dx}\left[\int_{101}^{x} f(t)dt\right] - \frac{d}{dx}\left[\operatorname{arcsec}(\ln x)\right] + \frac{d}{dx}\left[\int_{102}^{x} (t^2+1) f(t) dt\right]$$

$$0 = f(x) - \frac{1}{\ln x\sqrt{(\ln x)^2-1}}\,\frac{1}{x} + (x^2+1)f(x) \qquad \text{by } 2^{nd} \text{ FTC}$$

$$f(x) + (x^2+1)f(x) = \frac{1}{x\ln x\sqrt{(\ln x)^2-1}}$$

$$(x^2+2)f(x) = \frac{1}{x\ln x\sqrt{(\ln x)^2-1}}$$

$$f(x) = \frac{1}{x(x^2+2)\ln x\sqrt{(\ln x)^2-1}}$$

where $x \neq 0$, $x \neq 1$ and $(\ln x)^2 - 1 > 0$

$(\ln x)^2 > 1$

$-1 > \ln x > 1$

$e^{-1} > x > e$

$\therefore$ the domain of $f(x)$ is $(-\infty, \frac{1}{e}) \cup (e, \infty)$