**Dawson College: Calculus II**: 201-NYB-05-S02

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# Test 1

This test is graded out of 45 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

**Formula:**

$\sum_{i=1}^{n} c = cn$ where $c$ is a constant $\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

**Question 1.** *(5 marks)* Evaluate using the definition of the definite integral.

$$\int_1^2 3x^2 - 2x + 1\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$x_i = a + i\Delta x$

$x_i = 1 + \frac{i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[3\left(1+\frac{i}{n}\right)^2 - 2\left(1+\frac{i}{n}\right) + 1\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[3\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) - 2 - \frac{2i}{n} + 1\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[3 + \frac{6i}{n} + \frac{3i^2}{n^2} - 2 - \frac{2i}{n} + 1\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left[\frac{3i^2}{n^2} + \frac{4i}{n} + 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \left[\frac{3}{n^2}\sum_{i=1}^{n} i^2 + \frac{4}{n}\sum_{i=1}^{n} i + \sum_{i=1}^{n} 2\right]$$

$$= \lim_{n\to\infty} \frac{1}{\cancel{n}} \left[\frac{3}{n^2}\frac{\cancel{n}(n+1)(2n+1)}{6} + \frac{\cancel{4}^{\,2}}{n}\frac{\cancel{n}(n+1)}{\cancel{2}} + 2\cancel{n}\right]$$

$$= 2 + \lim_{n\to\infty}\left[\frac{3(n+1)(2n+1)}{6\cdot n\cdot n} + 2\frac{(n+1)}{n} + 2\right]$$

$$= 2 + \frac{3}{6}\cdot 1\cdot 2 + 2$$

$$= 5$$

**Question 2.** *(5 marks)* Evaluate the following definite integral:

$$\int_1^2 \sqrt{x}(x-1)(x-1)-\frac{1}{x}\,dx \quad = \int_1^2 x^{\frac{1}{2}}(x^2-2x+1)-\frac{1}{x}\,dx$$

$$= \int_1^2 x^{5/2}-2x^{3/2}+x^{1/2}-\frac{1}{x}\,dx$$

$$= \left[\frac{2x^{7/2}}{7}-\frac{4x^{5/2}}{5}+\frac{2x^{3/2}}{3}-\ln|x|\right]_1^2$$

$$= \left[\frac{2\cdot 2^{7/2}}{7}-\frac{4\cdot 2^{5/2}}{5}+\frac{2\,2^{3/2}}{3}-\ln|2|\right]-\left[\frac{2}{7}-\frac{4}{5}+\frac{2}{3}-\ln|1|\right]$$

$$= \frac{2\sqrt{128}}{7}-\frac{4\sqrt{32}}{5}+\frac{2\sqrt{8}}{3}-\ln 2-\frac{2}{7}+\frac{4}{5}-\frac{2}{3}$$

**Question 3.** *(5 marks)* Evaluate the following indefinite integral:

$$\int \frac{y^2}{\sqrt{y-1}}\,dy \quad = \int \frac{y^2}{\sqrt{u}}\,du$$

$u = y-1$

$du = dy$

$$= \int \frac{(u+1)^2}{\sqrt{u}}\,du$$

$$= \int (u^2+2u+1)u^{-\frac{1}{2}}\,du$$

$$= \int u^{3/2}+2u^{1/2}+u^{-1/2}\,du$$

$$= \frac{2u^{5/2}}{5}+\frac{4u^{3/2}}{3}+2u^{\frac{1}{2}}+C$$

$$= \frac{2(y-1)^{5/2}}{5}+\frac{4(y-1)^{3/2}}{3}+2(y-1)^{1/2}+C$$

**Question 4.** *(5 marks)* Find the average value of the function

$$f(x) = \frac{x^2}{x^2+3}$$

on the interval $[0, \sqrt{3}]$

$$\text{average value of function} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{\sqrt{3}-0}\int_0^{\sqrt{3}} \frac{x^2+3-3}{x^2+3}\,dx$$

$$= \frac{1}{\sqrt{3}}\left[\int_0^{\sqrt{3}} 1\,dx - 3\int_0^{\sqrt{3}} \frac{1}{x^2+3}\,dx\right]$$

$$= \frac{1}{\sqrt{3}}\left[\left[x\right]_0^{\sqrt{3}} - 3\left[\frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}}\right]_0^{\sqrt{3}}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\sqrt{3}-0-3\left[\frac{1}{\sqrt{3}}\arctan\frac{\sqrt{3}}{\sqrt{3}} - \underbrace{\frac{1}{\sqrt{3}}\arctan\frac{0}{\sqrt{3}}}_{=0}\right]\right]$$

$$= \frac{1}{\sqrt{3}}\left[\sqrt{3} - \frac{3}{\sqrt{3}}\frac{\pi}{4}\right] = 1 - \frac{\pi}{4} = \frac{4-\pi}{4}$$

**Question 5.** *(5 marks + 1 bonus mark to simplify completely)* Evaluate the following expression:

$$\frac{d}{dx}\left[\int_{-\cos x}^{\cos x} t^2 \arctan t\,dt\right]$$

method 1: notice $f(x) = x^2\arctan x$ is odd since

$$f(-x) = (-x)^2\arctan(-x)$$
$$= x^2(-\arctan x)$$
$$= -x^2\arctan x$$
$$= -f(x)$$

$$\therefore \frac{d}{dx}\left[\int_{-\cos x}^{\cos x} t^2\arctan t\,dt\right] = \frac{d}{dx}\left[0\right] = 0$$

method 2: $g(x) = \int_{-\cos x}^{\cos x} t^2\arctan t\,dt$

$$= \int_{-\cos x}^{0} t^2\arctan t\,dt + \int_0^{\cos x} t^2\arctan t\,dt$$

$$= -\int_0^{-\cos x} t^2\arctan t\,dt + \int_0^{\cos x} t^2\arctan t\,dt$$

$$= -f(g_1(x)) + f(g_2(x))$$

where $f(x) = \int_0^x t^2\arctan t\,dt$

$f'(x) = x^2\arctan x$ by 2nd FTC

$$\therefore g'(x) = -f'(g_1(x))g_1'(x) + f'(g_2(x))g_2'(x)$$

$$= -(-\cos x)^2\arctan(-\cos x)\sin x + (\cos x)^2\arctan(\cos x)(-\sin x)$$

$$= (\cos x)^2\arctan(\cos x)\sin x - (\cos x)^2\arctan(\cos x)\sin x$$

$= 0$ since $\arctan x$ is odd.

$g_1(x) = -\cos x$
$g_1'(x) = \sin x$
$g_2(x) = \cos x$
$g_2'(x) = -\sin x$

**Question 6.** *(5 marks)* Suppose $f(x)$ is continous over the real numbers and

$$\int_0^1 f(x)\,dx = 121.$$

Evaluate

$$\int_{-\pi}^{0} \sin\left(\frac{x}{2}\right) f\left(\cos\left(\frac{x}{2}\right)\right)\,dx. \quad = \int_0^1 f(u)(-2du) \quad = -2\int_0^1 f(u)du$$

$$= -2(121)$$

$$= -242$$

$u = \cos\left(\frac{x}{2}\right)$

$du = -\sin\left(\frac{x}{2}\right)\frac{1}{2}dx$

$-2\,du = \sin\left(\frac{x}{2}\right)dx$

$u(0) = \cos\left(\frac{0}{2}\right) = 1$

$u(-\pi) = \cos\left(\frac{-\pi}{2}\right) = 0$

**Question 7.** *(5 marks)* Evaluate the following definite integral.

$$\int x \operatorname{arcsec} x\,dx \quad = uv - \int v\,du$$

$u = \operatorname{arcsec} x \qquad du = \frac{1}{x\sqrt{x^2-1}}dx$

$v = \frac{x^2}{2} \qquad dv = x\,dx$

$$= \frac{x^2 \operatorname{arcsec} x}{2} - \int \frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}\sqrt{x^2-1}}dx$$

$$= \frac{x^2 \operatorname{arcsec} x}{2} - \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}dx$$

$u = x^2 - 1$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \frac{x^2 \operatorname{arcsec} x}{2} - \frac{1}{2}\int \frac{1}{\sqrt{u}}\frac{du}{2}$$

$$= \frac{x^2 \operatorname{arcsec} x}{2} - \frac{1}{4}\int (u)^{-\frac{1}{2}}du$$

$$= \frac{x^2 \operatorname{arcsec} x}{2} - \frac{1}{4}\left[2u^{\frac{1}{2}}\right] + C$$

$$= \frac{x^2 \operatorname{arcsec} x}{2} - \frac{1}{2}\sqrt{x^2-1} + C$$

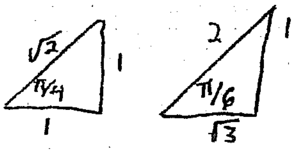
**Question 8.** *(5 marks)* Evaluate the following definite integral:

$$\int_{\pi/18}^{\pi/12} t\csc 3t\cot 3t\,dt = \left[-\frac{t\csc 3t}{3}\right]_{\pi/18}^{\pi/12} - \int_{\frac{\pi}{18}}^{\pi/12} -\frac{\csc 3t}{3}\,dt$$

$u = t \qquad du = dt$

$v = -\frac{\csc 3t}{3} \qquad dv = \csc 3t\cot 3t\,dt$

$$= \frac{-\frac{\pi}{12}\csc 3\left(\frac{\pi}{12}\right)}{3} + \frac{\frac{\pi}{18}\csc 3\left(\frac{\pi}{18}\right)}{3} + \frac{1}{3}\int_{\pi/18}^{\pi/12}\csc 3t\,dt$$




$$= -\frac{\pi\csc\frac{\pi}{4}}{36} + \frac{\pi\csc\frac{\pi}{6}}{54} - \frac{1}{9}\Big[\ln\left|\csc 3t + \cot 3t\right|\Big]_{\frac{\pi}{18}}^{\pi/12}$$

$$= \frac{-\pi\sqrt{2}}{36} + \frac{2\pi}{54} - \frac{1}{9}\ln\left|\csc 3\left(\tfrac{\pi}{12}\right) + \cot 3\left(\tfrac{\pi}{12}\right)\right| + \frac{1}{9}\ln\left|\csc 3\left(\tfrac{\pi}{18}\right) + \cot 3\left(\tfrac{\pi}{18}\right)\right|$$

$$= \frac{\pi}{27} - \frac{\pi\sqrt{2}}{36} - \frac{1}{9}\ln\left|\sqrt{2}+1\right| + \frac{1}{9}\ln\left|2+\sqrt{3}\right| = \frac{\pi}{27} - \frac{\pi\sqrt{2}}{36} + \ln\sqrt[9]{\frac{2+\sqrt{3}}{1+\sqrt{2}}}$$

**Question 9.** *(5 marks)* Prove: If $f(x)$ is a continuous and even function then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(-x)dx + \int_{0}^{a} f(x)dx$$

since $f(x)$ is even

$u = -x$

$du = -dx$

$u(0) = -0 = 0$

$u(-a) = -(-a) = a$

$$= -\int_{0}^{a} f(u)(-du) + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(u)\,du + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

**Bonus Question.** *(3 marks)*
If $f(x)$ is a continuous function on a certain domain and satisfies

$$0 = \int_{98}^{x} f(t)\,dt - \arcsin(\ln x) - \int_{x}^{99} e^t f(t)\,dt$$

then find $f(x)$ and state its domain.

$$0 = \int_{98}^{x} f(t)\,dt - \arcsin(\ln x) + \int_{99}^{x} e^t f(t)\,dt$$

$$\frac{d}{dx}[0] = \frac{d}{dx}\left[\int_{98}^{x} f(t)\,dt\right] - \frac{d}{dx}\left[\arcsin(\ln x)\right] + \frac{d}{dx}\left[\int_{99}^{x} e^t f(t)\,db\right]$$

$$0 = f(x) - \frac{1}{\sqrt{1-(\ln x)^2}}\,\frac{1}{x} + e^x f(x)$$

by $2^{nd}$ FTC

$$f(x) + e^x f(x) = \frac{1}{\sqrt{1-(\ln x)^2}}\,\frac{1}{x}$$

$$f(x)(1+e^x) = \frac{1}{\sqrt{1-(\ln x)^2}}\,\frac{1}{x}$$

$$f(x) = \frac{1}{x(1+e^x)\sqrt{1-(\ln x)^2}}$$

So $x \neq 0$ and $1-(\ x)^2 > 0$

$$1 > (\ln x)^2$$

$$1 > \ln x > -1$$

$$e > x > e^{-1}$$

$\therefore$ the domain is $(\frac{1}{e}, e)$