Dawson College: **Calculus II**: 201-NYB-05-S02: Winter 2010

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Test 2

This test is graded out of ~~43~~ 48 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(3 marks)* Evaluate the definite integral:

$$\int_{\pi/4}^{\pi/3} \cos^2 z\, dz. = \int_{\pi/4}^{\pi/3} \frac{1+\cos 2z}{2}\, dz = \left[\frac{1}{2} + \frac{1}{4}\sin 2z\right]_{\pi/4}^{\pi/3} = \left[\frac{1}{2}z + \frac{1}{2}\sin z\cos z\right]_{\pi/4}^{\pi/3}$$

$$= \left[\frac{1}{2}\left(\frac{\pi}{3}\right) + \frac{1}{2}\sin\frac{\pi}{3}\cos\frac{\pi}{3}\right] - \left[\frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{2}\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right]$$

$$= \frac{\pi}{6} + \frac{1}{2}\frac{\sqrt{3}}{2}\frac{1}{2} - \left[\frac{\pi}{8} + \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right]$$

$$= \frac{\pi}{6} - \frac{\pi}{8} + \frac{\sqrt{3}}{8} - \frac{1}{4}$$

Question 2. *(5 marks)* Evaluate the improper integral or show it diverges:

$$\int_5^\infty \frac{1}{(x-3)\sqrt{x^2-6x+5}}\, dx = \int_5^\infty \frac{1}{(x-3)\sqrt{(x-3)^2-4}}\, dx = \int_5^6 \frac{1}{(x-3)\sqrt{(x-3)^2-4}}\, dx + \int_6^\infty \frac{1}{(x-3)\sqrt{(x-3)^2-4}}\, dx$$

notice:
x^2-6x+5
$= x^2-6x+9-9+5$
$= (x-3)^2-4$

$$= \lim_{a\to 5^+} \int_a^6 \frac{1}{(x-3)\sqrt{(x-3)^2-4}}\, dx + \lim_{b\to\infty} \int_6^b \frac{1}{(x-3)\sqrt{(x-3)^2-4}}\, dx$$

$$= \lim_{a\to 5^+} \left[\frac{1}{2}\operatorname{arcsec}\frac{x-3}{2}\right]_a^6 + \lim_{b\to\infty} \left[\frac{1}{2}\operatorname{arcsec}\frac{x-3}{2}\right]_6^b$$

$$= \frac{1}{2}\operatorname{arcsec}\frac{6-3}{2} - \underbrace{\lim_{a\to 5^+}\frac{1}{2}\operatorname{arcsec}\frac{a-3}{2}}_{0} + \underbrace{\lim_{b\to\infty}\frac{1}{2}\operatorname{arcsec}\frac{b-3}{2}}_{\pi/2} - \frac{1}{2}\operatorname{arcsec}\frac{6-3}{2}$$

$$= \frac{\pi}{2}$$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int \csc^3 2\theta \cot^3 2\theta \, d\theta = \int \csc^2 2\theta \cot^2 2\theta \csc 2\theta \cot 2\theta \, d\theta$$

$u = \csc 2\theta$

$du = -2\csc 2\theta \cot 2\theta \, d\theta$

$\frac{-1}{2} du = \csc 2\theta \cot 2\theta \, d\theta$

$$= \int \csc^2 2\theta \left(\csc^2 2\theta - 1 \right) \csc 2\theta \cot 2\theta \, d\theta$$

$$= \int u^2 (u^2 - 1) \left(\frac{-du}{2} \right)$$

$$= \frac{1}{2} \int u^2 - u^4 \, du$$

$$= \frac{1}{2} \left[\frac{u^3}{3} - \frac{u^5}{5} \right] + C$$

$$= \frac{\csc^3 2\theta}{6} - \frac{\csc^5 2\theta}{10} + C$$

Question 4. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{\sqrt{4x^2 - 9}}{x^4} dx = \int \frac{\sqrt{(2x)^2 - 3^2}}{X^4} dx = \int \frac{\sqrt{(3\sec\theta)^2 - 3^2}}{\left(\frac{3}{2}\sec\theta\right)^4} \frac{3}{2}\sec\theta \tan\theta \, d\theta$$

$2x = 3\sec\theta$

$2dx = 3\sec\theta \tan\theta \, d\theta$

$dx = \frac{3}{2} \sec\theta \tan\theta \, d\theta$

$X = \frac{3}{2} \sec\theta$

$$= \frac{\cancel{3}}{2\left(\frac{3}{2}\right)^{\cancel{4}\,3}} \int \frac{\sqrt{9\sec^2\theta - 9}}{\sec^{\cancel{4}\,3}\theta} \cancel{\sec\theta} \tan\theta \, d\theta$$

$$= \frac{8}{27} \int \frac{\sqrt{9(\sec^2\theta - 1)}}{\sec^3\theta} \tan\theta \, d\theta$$

$$= \frac{8}{27} \int \frac{\sqrt{9\tan^2\theta} \tan\theta \, d\theta}{\sec^3\theta}$$

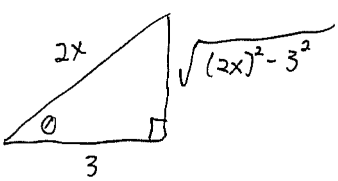
$$= \frac{8}{9} \int \frac{\tan^2\theta \, d\theta}{\sec^3\theta}$$

$\frac{\text{hyp.}}{\text{adj}} = \frac{2x}{3} = \sec\theta$

$$= \frac{8}{9} \int \cos^{\cancel{3}}\theta \frac{\sin^2\theta \, d\theta}{\cancel{\cos^2\theta}}$$

$u = \sin\theta$

$du = \cos\theta \, d\theta$



$$= \frac{8}{9} \int u^2 du$$

$$= \frac{8}{9} \frac{u^3}{3} + C$$

$$= \frac{8}{27} \sin^3\theta + C = \frac{8}{27} \left(\frac{\sqrt{(2x)^2 - 3^2}}{2x} \right)^3 + C$$

Question 5. *(5 marks)* Sketch and find the total area of the region(s) bounded by the graphs of $y = \frac{90}{x^2+9}$, $y = 10 - x$.

Sketch of $y = \frac{90}{x^2+9}$ notice: $\lim_{x \to \pm\infty} \frac{90}{x^2+9} = 0$

Local min/max: $y' = \frac{-90}{(x^2+90)}(2x)$

y-int: @ $x = 0$ $y = \frac{90}{9} = 10$

$\therefore$ $x = 0$ critical point a local max.

Sketch of $y = 10 - x$: x-int: $(10, 0)$
y-int: $(0, 10)$

Intersection of two curves: $\frac{90}{x^2+9} = 10 - x$

$90 = (10 - x)(x^2+9)$

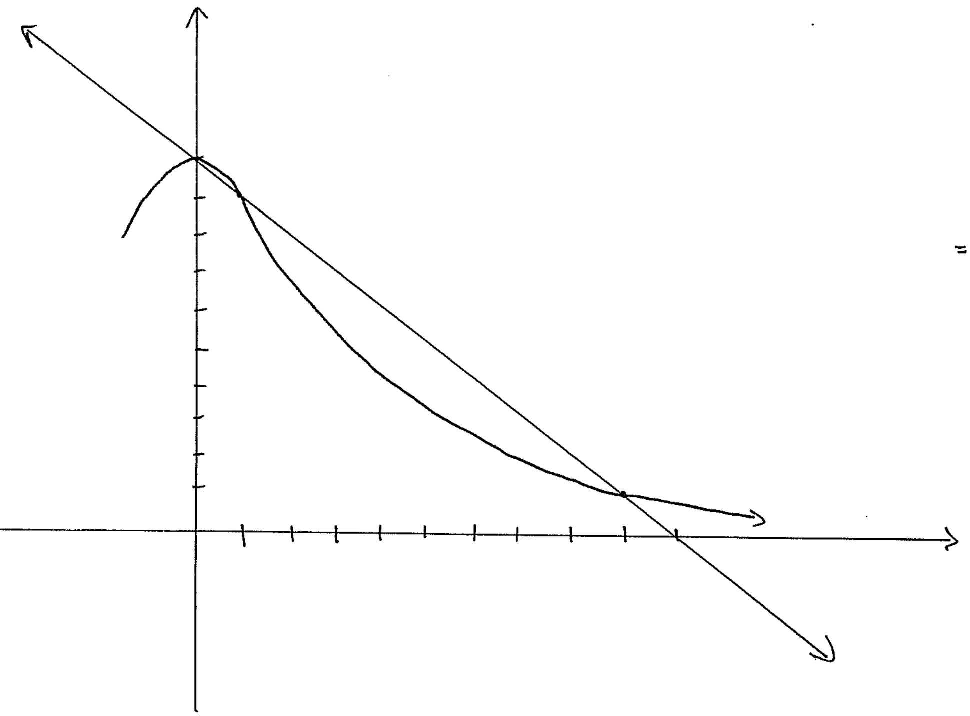
$90 = 10x^2 - x^3 + 90 - 9x$

$x^3 - 10x^2 + 9x = 0$

$x(x^2 - 10x + 9) = 0$

$x(x-1)(x-9) = 0$

$x = 0$ $x = 1$ $x = 9$



$$\text{Area} = \int_0^1 \frac{90}{x^2+9} - (10 - x)\, dx + \int_1^9 (10 - x) - \frac{90}{x^2+9}\, dx$$

$$= \left[\frac{90}{3}\arctan\frac{x}{3} - 10x + \frac{x^2}{2}\right]_0^1 + \left[10x - \frac{x^2}{2} - \frac{90}{3}\arctan\frac{x}{3}\right]_1^9$$

$$= 30\arctan\frac{1}{3} - 10 + \frac{1}{2}$$

$$+ 90 - \frac{81}{2} - 30\arctan 3$$

$$-10 + \frac{1}{2} + 30\arctan\frac{1}{3}$$

$$= 60\arctan\frac{1}{3} + 70 - \frac{79}{2} - 30\arctan 3$$

Question 6. *(5 marks)* Evaluate the limit:

$$\lim_{x\to 3^-} (\ln(4-x))^{3-x}$$

I.F. 0^0

Let $y = \lim_{x\to 3^-} (\ln(4-x))^{3-x}$

$\ln y = \ln\left(\lim_{x\to 3^-} (\ln(4-x))^{3-x}\right)$

$\ln y = \lim_{x\to 3^-} \ln(\ln(4-x))^{3-x}$

$\ln y = \lim_{x\to 3^-} (3-x)\ln(\ln(4-x))$ I.F. $0 \cdot -\infty$

$\ln y = \lim_{x\to 3^-} \dfrac{\ln(\ln(4-x))}{\frac{1}{3-x}}$ I.F. $\frac{-\infty}{\infty}$

$\ln y = \lim_{x\to 3^-} \dfrac{\frac{1}{\ln(4-x)}\frac{1}{(4-x)}(-1)}{\frac{-1}{(3-x)^2}(-1)}$ by $\hat{H}$

$\ln y = \lim_{x\to 3^-} \dfrac{-(3-x)^2}{(4-x)\ln(4-x)}$ I.F. $\frac{0}{0}$

$\ln y = \lim_{x\to 3^-} \dfrac{-2(3-x)(-1)}{-\ln(4-x) + \frac{(4-x)}{(4-x)}(-1)}$ by $\hat{H}$

$\ln y = 0$

$y = e^0 = 1$

Question 7. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{1}{x^3-4x}\,dx = \int \frac{1}{x(x^2-4)}\,dx = \int \frac{1}{x(x-2)(x+2)}\,dx$$

$$\frac{1}{x(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2}$$

$$1 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$$

Let $x=0$

$$1 = A(0-2)(0+2) + B(0)(0+2) + C(0)(0-2)$$
$$1 = -4A$$
$$\frac{-1}{4} = A$$

Let $x=2$

$$1 = A(2-2)(2+2) + B(2)(2+2) + C(2)(2-2)$$
$$1 = 8B$$
$$\frac{1}{8} = B$$

Let $x=-2$

$$1 = A(-2-2)(-2+2) + B(-2)(-2+2) + C(-2)(-2-2)$$
$$1 = 8B$$
$$\frac{1}{8} = B$$

$$\therefore \int \frac{1}{x(x-2)(x+2)}\,dx = \int \frac{-1}{4}\frac{1}{x} + \frac{1}{8}\frac{1}{x-2} + \frac{1}{8}\frac{1}{x+2}\,dx$$

$$= \frac{-1}{4}\ln|x| + \frac{1}{8}\ln|x-2| + \frac{1}{8}\ln|x+2| + C$$

Question 8. Only answer one of the following two questions.

a. *(5 marks + 1 bonus mark for complete and correct solution)* Evaluate the limit

$$\lim_{x\to 0} \frac{\int_0^{-x} \tan 2t\, dt + \int_1^{\cos x} \csc t\, dt}{\cos ax - \cos bx}$$

where $a, b \neq 0$ and $a \neq b$.

b. *(5 marks)* Find the value(s) of a and b such that

$$\lim_{x\to 0} \left(\frac{\ln(1-x)}{x^2} + a + \frac{b}{x} \right) = 0$$

a) $\lim_{x\to 0} \dfrac{\int_0^{-x} \tan 2t\,dt + \int_1^{\cos x} \csc t\,dt}{\cos ax - \cos bx}$ I.F. $\frac{0}{0}$

$= \lim_{x\to 0} \dfrac{-(\tan -2x) + \csc(\cos x)(-1)\sin x}{-a\sin ax + b\sin bx}$ by $\hat{H}$ and 2^{nd} FTC I.F $\frac{0}{0}$

$= \lim_{x\to 0} \dfrac{-\sec^2(2x)(-2) - [-\csc(\cos x)\cot(\cos x)(-\sin x)\sin x + \cos x \csc(\cos x)]}{-a^2\cos ax + b^2 \sin bx}$ by $\hat{H}$

$= \dfrac{2 + \csc 1}{b^2 - a^2}$

b) $0 = \lim_{x\to 0} \left(\dfrac{\ln(1-x)}{x^2} + a + \dfrac{b}{x} \right)$

$0 = \lim_{x\to 0} \left(\dfrac{\ln(1-x) + ax^2 + bx}{x^2} \right)$ I.F $\frac{0}{0}$

$0 = \lim_{x\to 0} \left(\dfrac{\frac{1}{1-x}(-1) + 2ax + b}{2x} \right)$ by $\hat{H}$ need I.F. $\frac{0}{0}$ $\therefore b = 1$

$0 = \lim_{x\to 0} \left(\dfrac{\frac{-1}{(1-x)^2} + 2a}{2} \right)$ by $\hat{H}$

$0 = \dfrac{-1 + 2a}{2}$

$\therefore a = \dfrac{+1}{2}$

Question 9. Only answer one of the following two questions.

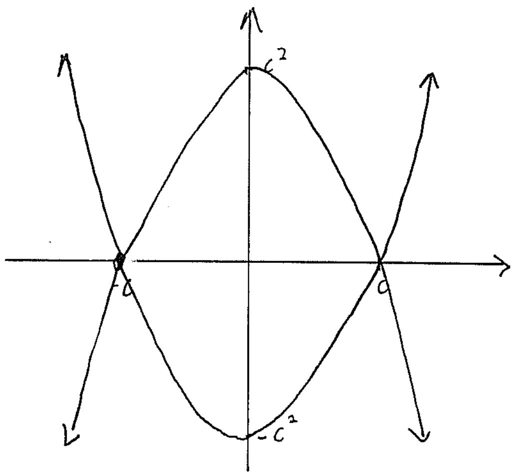
a. *(5 marks)* Find the value(s) of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 1944. Sketch the graphs of the functions.

b. *(5 marks)* Find the value of a such that the line $x = a$ bisects the area under the curve $y = 1/x^5$ on $1 \le x < \infty$. Sketch the graphs of the function and relation.

a) Lets find the intersection of both curves

$$x^2 - c^2 = c^2 - x^2$$
$$2x^2 - 2c^2 = 0$$
$$x^2 - c^2 = 0$$
$$(x-c)(x+c) = 0$$
$$x = c \qquad x = -c$$

the x-int. of both curves are $x = \pm c$, the vertex of $y = x^2 - c^2$ $(0, -c^2)$
$y = -x^2 + c^2$ $(0, c^2)$

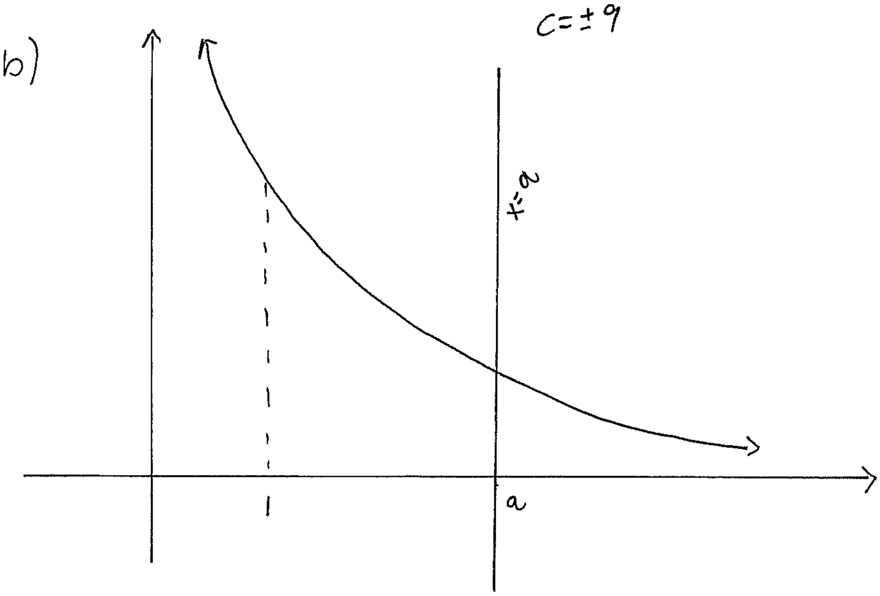


$$\text{Area} = \int_{-c}^{c} c^2 - x^2 - (x^2 - c^2)\,dx$$
$$= \int_{-c}^{c} 2c^2 - 2x^2\,dx$$
$$= \left[2c^2x - \frac{2x^3}{3}\right]_{-c}^{c}$$
$$= 2c^2c - \frac{2c^3}{3} - \left[2c^2(-c) - \frac{2}{3}(-c)^3\right]$$
$$= \frac{8c^3}{3}$$

$\therefore\ 1944 = \frac{8}{3}c^3$

$9 = c$ But notice $c = \pm 9$

b)



$$\int_1^a \frac{1}{x^5}\,dx = \int_a^\infty \frac{1}{x^5}\,dx$$
$$\left[\frac{1}{-4\cancel{x}^4}\right]_1^a = \lim_{b\to\infty} \int_a^b \frac{1}{x^5}\,dx$$
$$\frac{1}{4} - \frac{1}{4a^4} = \lim_{b\to\infty} \left[\frac{1}{-4x^4}\right]_a^b$$
$$\frac{1}{4} - \frac{1}{4a^4} = \lim_{b\to\infty} \left[\frac{1}{4a^4} - \cancel{\frac{1}{4b^4}}^{\,0}\right]$$
$$\frac{1}{4} = \frac{2}{4a^4}$$
$$a = \sqrt[4]{2}$$

Question 10. Only answer one of the following two questions.

a. *(5 marks + 2 bonus marks for complete and correct solution)* If $f(x)$ is a quartic function such that $f(0) = 4$, $f''(0) = 18$ and

$$\int \frac{f(x)}{x^2(x-1)^2(x^2+1)}\,dx$$

is a rational function, find the value of $f^{(4)}(x)$.

b. *(5 marks)* Find the value of the constant C for which the integral

$$\int_0^{\infty}\left(\frac{2x}{3x^2+1}-\frac{C}{3x+1}\right)dx$$

converges. Evaluate the integral for this value of C.

Since $f(x)$ is a quartic function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f(0) = 4 \iff e = 4$

$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 12ax^2 + 6bx + 2c$

$f''(0) = 18 \iff c = 9$

$$\frac{ax^4 + bx^3 + 9x^2 + dx + 4}{x^2(x-1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}$$

Since integral is rational $\Rightarrow A = 0,\ C = 0,\ E = 0,\ F = 0$

$ax^4 + bx^3 + 9x^2 + dx + 4 = B(x-1)^2(x^2+1) + Dx^2(x^2+1)$

$ax^4 + bx^3 + 9x^2 + dx + 4 = (B+D)x^4 - 2Bx^3 + Dx^2 - 2Bx + B$

$\therefore B = 4$ and $D = 9$

$\therefore a = B + D = 13$

and $f'''(x) = 24ax + 6b$, $f^{(4)}(x) = 24a$, $\therefore f^{(4)}(x) = 24(13)$

b) $$\int_0^{\infty} \frac{2x}{3x^2+1} - \frac{C}{3x+1}\,dx = \lim_{b\to\infty} \int_0^b \frac{2x}{3x^2+1} - C\frac{1}{3x+1}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{1}{3}\ln 3x^2+1 - \frac{C}{3}\ln|3x+1|\right]_0^b$$

$$= \lim_{b\to\infty} \ln\sqrt[3]{3b^2+1} - \ln\sqrt[3]{(3b+1)^C}$$

$$= \lim_{b\to\infty} \ln\sqrt[3]{\frac{3b^2+1}{(3b+1)^C}}$$

if $C > 2$ then limit tends to $-\infty$

if $C < 2$ " " " " ∞

if $C = 2$ then it converges

let $C = 2$

$$= \lim_{b\to\infty} \ln\sqrt[3]{\frac{3b^2+1}{9b^2+6b+1}}$$

$$= \ln\sqrt[3]{\frac{1}{3}}$$

Bonus Question. Only answer one of the following two bonus questions.

a. *(5 marks)* Show that if $f(x)$ is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of

$$\int_a^b f(x)\,dx$$

b. *(2 marks)* Show that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$$

a) It is sufficient to show that Simpson's Rule gives the exact value on two sub interval x_{i-1}, x_i, x_{i+1} where $\Delta x = h = \frac{b-a}{n}$

$$\underset{x_i - h}{\underset{\parallel}{x_{i-1}}}, \quad x_i, \quad \underset{x_i + h}{\underset{\parallel}{x_{i+1}}}$$

Let $f(x) = ax^3 + bx^2 + cx + d$

$$\int_{x_i-h}^{x_i+h} ax^3 + bx^2 + cx + d\,dx = \int_{-h}^{h} a(u+x_i)^3 + b(u+x_i)^2 + c(u+x_i) + d\,dx$$

let $u = x - x_i \Leftrightarrow u + x_i = x$

$du = dx$

$u(x_i+h) = x_i + h - x_i = h$

$u(x_i-h) = x_i - h - x_i = -h$

$$= \int_{-h}^{h} Au^3 + Bu^2 + Cu + D\,du$$

where $A = a$ Let $g(x) = Ax^3 + Bx^2 + Cx + D$

$B = b + 3ax_i$

$C = 2bx_i + 3ax_i^2 + c$

$D = cx_i + d + bx_i^2 + ax_i^3$

$$= \int_{-h}^{h} \underbrace{Au^3 + Cu}_{\text{odd}}\,du + \int_{-h}^{h} \underbrace{Bu^2 + D}_{\text{even}}\,du$$

$$= 0 + 2\int_0^h Bu^2 + D\,du$$

$$= 2\left[B\frac{u^3}{3} + Dx\right]_0^h = \frac{h}{3}(2Bh^2 + 6D)$$

$$= \frac{h}{3}\left(g(-h) + 4g(0) + g(h)\right)$$

$$= \frac{h}{3}\left(f(x_{i-1}) + 4f(x_i) + f(x_{i+1})\right)$$

Simpson's Rule

$\therefore$ Gives exact value on two sub interval.

Bonus Question. Only answer one of the following two bonus questions.

a . *(5 marks)* Show that if $f(x)$ is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of

$$\int_a^b f(x)\,dx$$

b . *(2 marks)* Show that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$$

Let $\displaystyle\int_a^b f(x)\,dx \approx \frac{1}{3}\left[\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right]\right] + \frac{2}{3}\left[\Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right]\right]$

$\Delta x = \dfrac{b-a}{n}$

$x_i = a + i\Delta x$

$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$= \frac{1}{2}\left(a + \frac{(2i-1)\Delta x}{2}\right)$

$$= \frac{1}{3}\left[\frac{\Delta x}{2}\left[f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n)\right]\right] + \frac{2}{3}\left[\Delta x\left[\sum_{i=1}^{n} f(\bar{x}_i)\right]\right]$$

$$= \frac{\Delta x}{2\cdot 3}\left[f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(\bar{x}_i)\right]$$

$$= \frac{\Delta x}{2\cdot 3}\left[f(x_0) + 4f(\bar{x}_1) + 2f(x_1) + 4f(\bar{x}_2) + \dots + 2f(x_{n-1}) + 4f(\bar{x}_n) + f(x_n)\right]$$

$$= \frac{\Delta x'}{3}\left[f(x_0') + 4f(x_1') + 2f(x_2') + \dots + 2f(x_{2n-2}') + 4(x_{2n-1}') + f(x_{2n}')\right]$$

where $\Delta x' = \dfrac{b-a}{2n} = \dfrac{\Delta x}{2}$

$x_i' = a + i\dfrac{\Delta x}{2}$

$= S_{2n}$.