Dawson College: Calculus II: 201-NYB-05-S02: Winter 2010

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Test 2

This test is graded out of 48 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(3 marks)* Evaluate the definite integral:

$$\int_{\pi/4}^{\pi/3} \sin^2 z\, dz = \int_{\pi/4}^{\pi/3} \frac{1-\cos 2z}{2}\, dz = \left[\frac{1}{2}x - \frac{1}{4}\sin 2z\right]_{\pi/4}^{\pi/3}$$

$$= \left[\frac{1}{2}x - \frac{1}{2}\sin z\cos z\right]_{\pi/4}^{\pi/3}$$

$$= \left[\frac{1}{2}\frac{\pi}{3} - \frac{1}{2}\sin\frac{\pi}{3}\cos\frac{\pi}{3}\right] - \left[\frac{1}{2}\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{4}\cos\frac{\pi}{4}\right]$$

$$= \frac{\pi}{6} - \frac{\pi}{8} - \frac{1}{2}\frac{\sqrt{3}}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{6} - \frac{\pi}{8} - \frac{\sqrt{3}}{8} + \frac{1}{4}$$

Question 2. *(5 marks)* Evaluate the improper integral or show it diverges:

$$\int_5^\infty \frac{1}{(x-2)\sqrt{x^2-4x-5}}\,dx = \int_5^\infty \frac{1}{(x-2)\sqrt{(x-2)^2-9}}\,dx = \int^6 \frac{1}{(x-2)\sqrt{(x-2)^2-9}}\,dx + \int_6^\infty \frac{1}{(x-2)\sqrt{(x-2)^2-9}}\,dx$$

notice:

$$x^2-4x-5$$
$$= x^2-4x+4-4-5$$
$$= (x-2)^2-9$$

$$= \lim_{a\to 5^+}\int_a^6 \frac{1}{(x-2)\sqrt{(x-2)^2-9}}\,dx + \lim_{b\to\infty}\int_6^b \frac{1}{(x-2)\sqrt{(x-2)^2-9}}\,dx$$

$$= \lim_{a\to 5^+}\left[\frac{1}{3}\operatorname{arcsec}\frac{(x-2)}{3}\right]_a^6 + \lim_{b\to\infty}\left[\frac{1}{3}\operatorname{arcsec}\frac{x-2}{3}\right]_6^b$$

$$= \lim_{a\to 5^+}\frac{1}{3}\operatorname{arcsec}\frac{4}{3} - \frac{1}{3}\underbrace{\operatorname{arcsec}\frac{a-2}{3}}_{\to 0} + \lim_{b\to\infty}\frac{1}{3}\underbrace{\operatorname{arcsec}\frac{b-2}{3}}_{\to \pi/2} - \frac{1}{3}\operatorname{arcsec}\frac{4}{3}$$

$$= \frac{\pi}{2}$$

Question 3. *(5 marks)* Evaluate the indefinite integral:

$$\int \csc^4 2\theta \cot^2 2\theta \, d\theta = \int \cot^2 2\theta \csc^2 2\theta \csc^2 2\theta \, d\theta$$

$u = \cot 2\theta$

$du = -\csc^2 2\theta \, 2 \, d\theta$

$-\frac{du}{2} = \csc^2 2\theta \, d\theta$

$$= \int \cot^2 2\theta \,(1+\cot^2 2\theta) \csc^2 2\theta \, d\theta$$

$$= \int u^2 (1+u^2) \, \frac{-du}{2}$$

$$= \frac{-1}{2} \int u^2 + u^4 \, du$$

$$= \frac{-1}{2}\left[\frac{u^3}{3} + \frac{u^5}{5}\right] + C$$

$$= -\frac{u^3}{6} - \frac{u^5}{10} + C = -\frac{\cot^3 2\theta}{6} - \frac{\cot^5 2\theta}{10} + C$$

Question 4. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{\sqrt{4x^2-9}}{x^4}\,dx = \int \frac{\sqrt{(2x)^2 - 3^2}}{x^4}\,dx = \int \frac{\sqrt{(3\sec\theta)^2 - 3^2}}{\left(\frac{3}{2}\sec\theta\right)^4} \, \frac{3\sec\theta\tan\theta \, d\theta}{2}$$

$2x = 3\sec\theta$

$2dx = 3\sec\theta\tan\theta\, d\theta$

$x = \frac{3}{2}\sec\theta$

$$= \frac{\cancel{3}}{\frac{3^{\cancel{4}3}}{2^{3}}} \int \frac{\sqrt{9(\sec^2\theta - 1)}}{\sec^{\cancel{4}3}\theta} \, \frac{\cancel{\sec\theta}\tan\theta\, d\theta}{\cancel{2}}$$

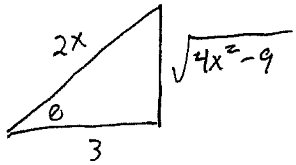
$$= \frac{2^3}{3^3} \int \frac{\sqrt{9\tan^2\theta}\,\tan\theta \, d\theta}{\sec^3\theta}$$

$$= \frac{2^3\,\cancel{3}}{3^{\cancel{3}2}} \int \frac{\tan^2\theta}{\sec^3\theta}\, d\theta$$

since $0 \le \theta < \frac{\pi}{2}$

$\frac{2x}{3} = \sec\theta$

$$= \frac{8}{9} \int \cos^3\theta \, \frac{\sin^2\theta}{\cancel{\cos^2\theta}}\, d\theta$$



$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{8}{9} \int \sin^2\theta \cos\theta \, d\theta$$

$$= \frac{8}{9} \int u^2 \, du$$

$$= \frac{8}{9} \, \frac{u^3}{3} + C$$

$$= \frac{8}{27} \sin^3\theta + C = \frac{8}{27}\left(\frac{\sqrt{4x^2-9}}{2x}\right)^3 + C$$

Question 5. *(5 marks)* Sketch and find the total area of the region(s) bounded by the graphs of $y = \frac{20}{x^2+4}$, $y = 5 - x$.

Sketch for $y = \frac{20}{x^2+4}$: notice: $\lim_{x \to \pm\infty} \frac{20}{x^2+4} = 0$

y-int: @ $x=0$ $y = \frac{20}{4} = 5$

local min/max: $y' = \frac{-20}{(x^2+4)^2} 2x$

$\therefore x=0$ critical point

a local max

Sketch for $y = 5-x$: y-int: $(0,5)$

x-int: $(5,0)$

Intersection of two curves:

$$\frac{20}{x^2+4} = 5-x$$

$$20 = (5-x)(x^2+4)$$

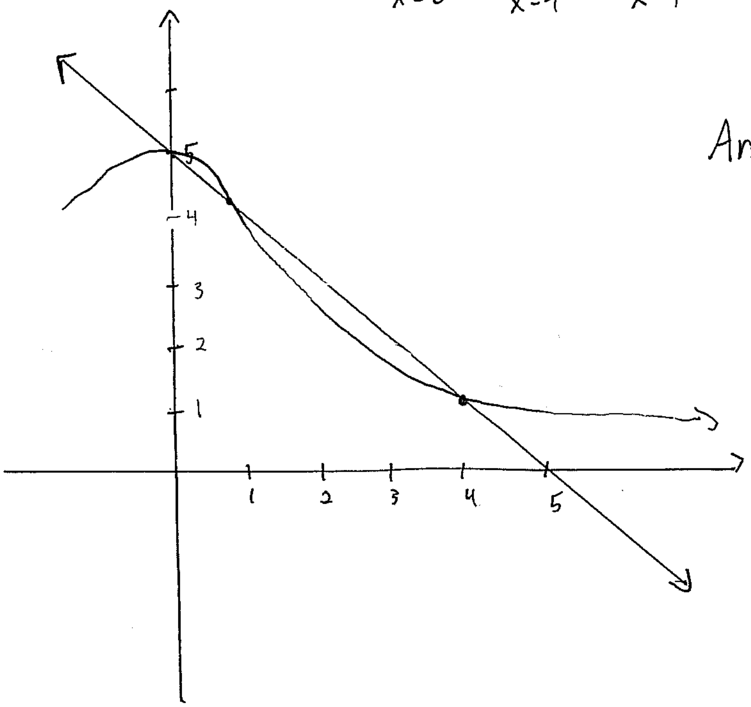
$$\cancel{20} = 5x^2 + \cancel{20} - x^3 - 4x$$

$$x^3 - 5x^2 + 4x = 0$$

$$x(x^2 - 5x + 4) = 0$$

$$x(x-4)(x-1) = 0$$

$x=0$ $x=4$ $x=1$



$$\text{Area} = \int_0^1 \frac{20}{x^2+4} - (5-x)\,dx$$

$$+ \int_1^4 5-x - \frac{20}{x^2+4}\,dx$$

$$= \left[\frac{20}{2}\arctan\frac{x}{2} - 5x + \frac{x^2}{2}\right]_0^1$$

$$+ \left[5x - \frac{x^2}{2} - \frac{20}{2}\arctan\frac{x}{2}\right]_1^4$$

$$= 10\arctan\frac{1}{2} - 5 + \frac{1}{2} + 20 - \frac{16}{2}$$

$$-10\arctan 2 - 5 + \frac{1}{2} + 10\arctan\frac{1}{2}$$

$$= 20\arctan\frac{1}{2} - 10\arctan 2 + 10 - 7$$

$$= 20\arctan\frac{1}{2} - 10\arctan 2 + 3$$

Question 6. *(5 marks)* Evaluate the limit:

$\lim_{x \to 2^-} (\ln(3-x))^{2-x}$ I.F. 0^0

Let $y = \lim_{x \to 2^-} (\ln(3-x))^{2-x}$

$\ln y = \ln \lim_{x \to 2^-} (\ln(3-x))^{2-x}$

$\ln y = \lim_{x \to 2^-} \ln (\ln(3-x))^{2-x}$

$\ln y = \lim_{x \to 2^-} (2-x) \ln(\ln(3-x))$ I.F. $0 \cdot -\infty$

$\ln y = \lim_{x \to 2^-} \frac{\ln(\ln(3-x))}{\frac{1}{(2-x)}}$ I.F. $\frac{-\infty}{\infty}$

$\ln y = \lim_{x \to 2^-} \frac{\frac{1}{\ln(3-x)} \frac{1}{(3-x)} (-1)}{\frac{1}{(2-x)^2}}$ by $\hat{H}$

$\ln y = \lim_{x \to 2^-} \frac{-(2-x)^2}{(3-x)\ln(3-x)}$ I.F. $\frac{0}{0}$

$\ln y = \lim_{x \to 2^-} \frac{-2(2-x)(-1)}{-\ln(3-x) + \frac{\cancel{(3-x)}(-1)}{\cancel{(3-x)}}}$ by $\hat{H}$

$\ln y = 0$

$y = e^0 = 1$

Question 7. *(5 marks)* Evaluate the indefinite integral:

$$\int \frac{1}{x^3-9x^2}\,dx = \int \frac{1}{x\,(x^2-9)}\,dx = \int \frac{1}{x\,(x-3)(x+3)}\,dx$$

$$\frac{1}{x\,(x-3)(x+3)} = \frac{B}{x} + \frac{C}{(x-3)} + \frac{D}{x+3}$$

$$1 = (B)(x-3)(x+3) + Cx\,(x+3) + Dx\,(x-3)$$

Let $x=0$

$$1 = (B)(0-3)(0+3) + C(0)(0+3) + D0(0-3)$$

$$1 = -9B$$

$$\frac{1}{-9} = B$$

Let $x=3$

$$1 = (B)(3-3)(3+3) + C(3)(3+3) + D(3)(3-3)$$

$$\frac{1}{18} = C$$

Let $x=-3$

$$1 = (B)(-3-3)(-3+3) + C(3)(-3+3) + D(-3)(-3-3)$$

$$\frac{+1}{18} = D$$

$$\therefore \int \frac{1}{x(x-3)(x+3)}\,dx = \int \frac{-1/9}{x} + \frac{1/}{x-3} + \frac{1/}{x+3}\,dx$$

$$= -\frac{1}{9}\ln|x| + \frac{1}{18}\ln|x-3| - \frac{1}{18}\ln|x+3| + C$$

Question 8. Only answer one of the following two questions.

a. *(5 marks + 1 bonus mark for complete and correct solution)* Evaluate the limit

$$\lim_{x\to 0} \frac{\int_e^{e-x} lnt\, dt + \int_0^{\sin x} \sec t\, dt}{\cos ax - \cos bx}$$

where $a, b \neq 0$ and $a \neq b$.

b. *(5 marks)* Find the value(s) of a and b such that

$$\lim_{x\to 0} \left(\frac{\tan 2x}{x^2} + a + \frac{b}{x} \right) = 0$$

a) $$\lim_{x\to 0} \frac{\int_e^{e-x} \ln t\, dt + \int_0^{\sin x} \sec t\, dt}{\cos ax - \cos bx} \quad \text{I.F. } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\ln(e-x)(-1) + \sec(\sin x)\cos x}{-a\sin ax + b\sin bx} \quad \text{by } \hat{H} \text{ and } 2^{nd} \text{ FTC} \quad \text{I.F. } \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\frac{1}{e-x} + \sec(\sin x)\tan(\sin x)\cos x - \sec(\sin x)\sin x}{-a^2\cos ax + b^2 \sin bx} \quad \text{by } \hat{H}$$

$$= \frac{\frac{1}{e} + 1}{b^2 - a^2}$$

b) $$0 = \lim_{x\to 0} \left(\frac{\tan 2x}{x^2} + a + \frac{b}{x} \right)$$

$$0 = \lim_{x\to 0} \left(\frac{ax^2 + bx + \tan 2x}{x^2} \right) \quad \text{I.F. } \frac{0}{0}$$

$$0 = \lim_{x\to 0} \left(\frac{a(2)x + b + \sec^2(2x)(2)}{2x} \right) \quad \text{by } \hat{H}$$

need I.F. $\frac{0}{0}$ $\therefore$ $b = -2$

$$0 = \lim_{x\to 0} \left(\frac{2a + 4\sec(2x)\sec(2x)\tan(2x)2}{2} \right) \quad \text{by } \hat{H}$$

$$0 = \frac{2a + 0}{2}$$

$\therefore$ $a = 0$

Question 9. Only answer one of the following two questions.

a. *(5 marks)* Find the value(s) of c such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 72. Sketch the graphs of the functions.

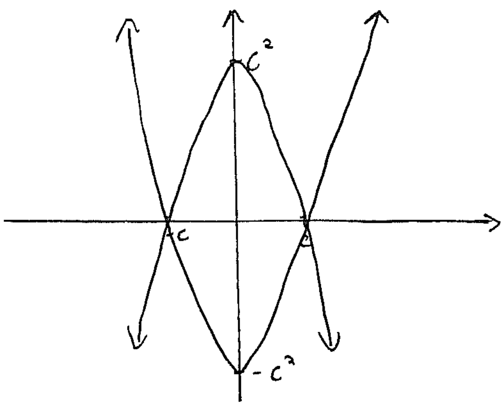
b. *(5 marks)* Find the value of a such that the line $x = a$ bisects the area under the curve $y = 1/x^3$ on $1 \le x < \infty$. Sketch the graphs of the function and relation.

a) lets find the intersection of both curves:

$$x^2 - c^2 = c^2 - x^2$$
$$2x^2 - 2c^2 = 0$$
$$x^2 - c^2 = 0$$
$$(x-c)(x+c) = 0$$
$$x = c \qquad x = -c$$

the x-int of both curves are $x = \pm c$, the vertex of $y = x^2 - c^2$ is $(0, -c^2)$

$y = -x^2 + c^2$ is $(0, c^2)$

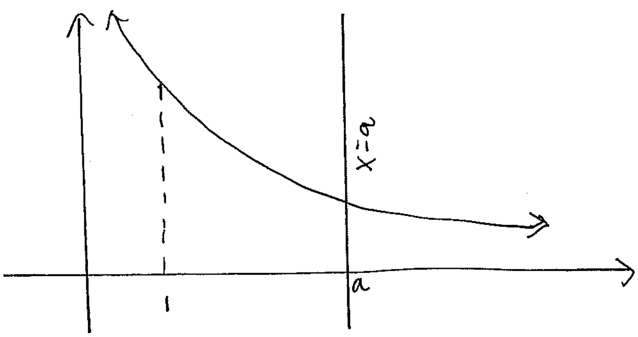


$$\text{Area} = \int_{-c}^{c} c^2 - x^2 - (x^2 - c^2)\,dx$$
$$= \int_{-c}^{c} 2c^2 - 2x^2\,dx$$
$$= \left[2c^2x - \frac{2x^3}{3}\right]_{-c}^{c}$$
$$= \left[2c^2c - \frac{2c^3}{3}\right] - \left[2c^2(-c) - \frac{2(-c)^3}{3}\right]$$
$$= \left[2c^3 - \frac{2}{3}c^3 + 2c^3 - \frac{2}{3}c^3\right]$$
$$= \frac{8}{3}c^3$$

$\therefore \quad 72 = \frac{8}{3}c^3$

$3. = c \qquad$ But notice $\quad c = \pm 3$

b)



$$\int_1^a \frac{1}{x^3}\,dx = \int_a^{\infty} \frac{1}{x^3}\,dx$$

$$\left[\frac{1}{-2x^2}\right]_1^a = \lim_{b \to \infty} \int_a^b \frac{1}{x^3}\,dx$$

$$\frac{-1}{2a^2} + \frac{1}{2} = \lim_{b \to \infty} \left[\frac{-1}{2x^2}\right]_a^b$$

$$\frac{1}{2} \quad \frac{-1}{2a^2} = \lim_{b \to \infty} \cancel{\frac{-1}{2b^2}}^{\,0} + \frac{1}{2a^2}$$

$$\frac{1}{2} = \frac{1}{a^2}$$

$$a = \sqrt{2}$$

Question 10. Only answer one of the following two questions

a. *(5 marks + 2 bonus marks for complete and correct solution)* If $f(x)$ is a quartic function such that $f(0) = 2$, $f''(0) = 12$ and

$$\int \frac{f(x)}{x^2(x+1)^2(x^2+1)}\,dx$$

is a rational function, find the value of $f^{(4)}(x)$.

b. *(5 marks)* Find the value of the constant C for which the integral

$$\int_0^{\infty}\left(\frac{x}{x^2+1} - \frac{C}{2x+1}\right)dx$$

converges. Evaluate the integral for this value of C.

a) Since $f(x)$ is a quartic function $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f(0) = 2 \iff e = 2$

$f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$f''(x) = 12ax^2 + 6bx + 2c$

$f''(0) = 12 \iff c = 6$

$$\frac{ax^4 + bx^3 + 6x^2 + dx + 2}{x^2(x+1)^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x+1)} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1}$$

Since integral is rational $\Rightarrow A = 0,\ C = 0,\ E = 0,\ F = 0$

$ax^4 + bx^3 + 6x^2 + dx + 2 = B(x+1)^2(x^2+1) + Dx^2(x^2+1)$

$ax^4 + bx^3 + 6x^2 + dx + 2 = (B+D)x^4 + 2Bx^3 + (2B+D)x^2 + 2Bx + B$

$\therefore B = 2$

$(2B+D) = 6 \Rightarrow D = 2$

$\therefore a = B + D = 4$ and $f'''(x) = 24ax + 6b$, $f^{(4)}(x) = 24(a) = 24(4) = 96$

b) $$\int_0^{\infty}\left(\frac{x}{x^2+1} - \frac{C}{2x+1}\right)dx = \lim_{b\to\infty}\int_0^b \frac{x}{x^2+1} - \frac{C}{2x+1}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{1}{2}\ln(x^2+1) - \frac{1}{2}C\ln|2x+1|\right]_0^b$$

$$= \lim_{b\to\infty}\left[\ln\sqrt{b^2+1} - \ln\sqrt{(2b+1)^C}\right]$$

$$= \lim_{b\to\infty}\ln\sqrt{\frac{b^2+1}{(2b+1)^C}}$$

if $C > 2$ then the limit tends to $-\infty$

if $C < 2$ " " " " " ∞

if $C = 2$ then it converges

let $C = 2$

$$= \lim_{b\to\infty}\ln\sqrt{\frac{b^2+1}{4b^2+4b+1}} = \ln\left(\frac{1}{2}\right)$$

Bonus Question. Only answer one of the following two bonus questions.

a. *(5 marks)* Show that if $f(x)$ is a polynomial of degree 3 or lower, then Simpson's Rule gives the exact value of

$$\int_a^b f(x)\,dx$$

b. *(2 marks)* Show that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$$

a) It is sufficient to show that Simpson's Rule gives the exact value on two sub interval x_{i-1}, x_i, x_{i+1} where $\Delta x = h = \frac{b-a}{n}$

$x_{i-1} = x_i - h$, $x_{i+1} = x_i + h$

Let $f(x) = ax^3 + bx^2 + cx + d$

$$\int_{x_i-h}^{x_i+h} ax^3+bx^2+cx+d\,dx = \int_{-h}^{h} a(u+x_i)^3 + b(u+x_i)^2 + c(u+x_i) + d\,dx$$

let $u = x - x_i \Leftrightarrow u + x_i = x$

$du = dx$

$u(x_i+h) = x_i + h - x_i = h$

$u(x_i-h) = x_i - h - x_i = -h$

$$= \int_{-h}^{h} Au^3 + Bu^2 + Cu + D\,du$$

where $A = a$

$B = b + 3a x_i$

$C = 2bx_i + 3a x_i^2 + c$

$D = cx_i + d + bx_i^2 + ax_i^3$

Let $g(x) = Ax^3 + Bx^2 + Cx + D$

$$= \underbrace{\int_{-h}^{h} Au^3 + Cu\,du}_{\text{odd}} + \underbrace{\int_{-h}^{h} Bu^2 + D\,du}_{\text{even}}$$

$$= 0 + 2\int_0^h Bu^2 + D\,du$$

$$= 2\left[B\frac{u^3}{3} + Dx\right]_0^h = \frac{h}{3}(2Bh^2 + 6D)$$

$$= \frac{h}{3}(g(-h) + 4g(0) + g(h))$$

$$= \frac{h}{3}(f(x_{i-1}) + 4f(x_i) + f(x_{i+1}))$$

Simpson's Rule

$\therefore$ Gives exact value on two sub interval.

Bonus Question. Only answer one of the following two bonus questions.

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$$\int_a^b f(x)\,dx$$

b . *(2 marks)* Show that

$$\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$$

Let $\int_a^b f(x)\,dx \approx \frac{1}{3}\left[\frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n)\right]\right] + \frac{2}{3}\left[\Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)\right]\right]$

$\Delta x = \frac{b-a}{n}$

$x_i = a + i\Delta x$

$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$

$= \frac{1}{2}\left(a + \frac{(2i-1)\Delta x}{2}\right)$

$$= \frac{1}{3}\left[\frac{\Delta x}{2}\left[f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n)\right]\right] + \frac{2}{3}\left[\Delta x\left[\sum_{i=1}^{n} f(\bar{x}_i)\right]\right]$$

$$= \frac{\Delta x}{2\cdot 3}\left[f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(\bar{x}_i)\right]$$

$$= \frac{\Delta x}{2\cdot 3}\left[f(x_0) + 4f(\bar{x}_1) + 2f(x_1) + 4f(\bar{x}_2) + \dots + 2f(x_{n-1}) + 4f(\bar{x}_n) + f(x_n)\right]$$

$$= \frac{\Delta x'}{3}\left[f(x_0') + 4f(x_1') + 2f(x_2') + \dots + 2f(x_{2n-2}') + 4(x_{2n-1}') + f(x_{2n}')\right]$$

where $\Delta x' = \frac{b-a}{2n} = \frac{\Delta x}{2}$

$x_i' = a + i\frac{\Delta x}{2}$

$= S_{2n}$.