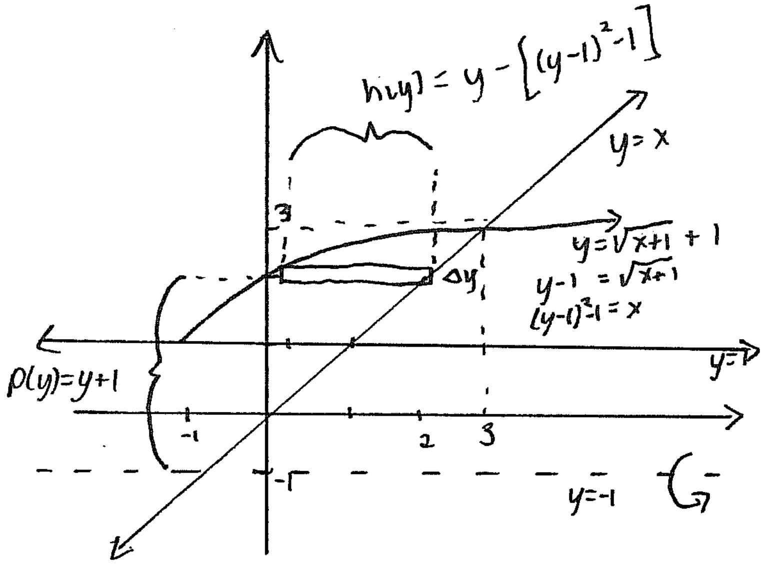
Name: Y. Lamontagne
Student ID:

Test 3

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = \sqrt{x+1}+1$, $y=x$ and $y=1$ rotated about the line $y=-1$.



intersection of $y=\sqrt{x+1}+1$ and $y=x$:

$\sqrt{x+1}+1 = x$

$\sqrt{x+1} = x-1$

$x+1 = (x-1)^2$

$x+1 = x^2-2x+1$

$0 = x^2-3x$

$0 = x(x-3)$

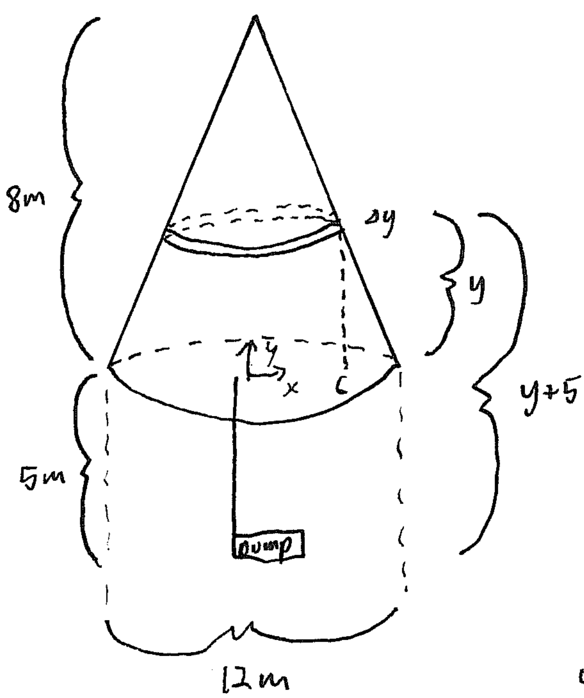
∴ volume of a shell

$\Delta V = 2\pi p(y)h(y)\Delta y$

$= 2\pi (y+1)(y-[(y-1)^2-1])\Delta y$

$$\therefore\ V = \int_1^3 2\pi (y+1)(y-[(y-1)^2-1])\,dy$$

Question 2. *(5 marks)* A tank which has the shape of a cone with its tip upwards has a $12m$ diameter on the bottom and a height of $8m$. Set up the integral to find the work required to fill the tank of a liquid with an arbitrary density ρ from a pump $5m$ below the tank.

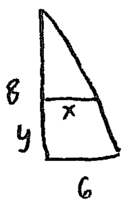


Volume of slice:

$\Delta V = \pi r^2 \Delta y$

$= \pi (6-x)^2 \Delta y$

$= \pi (6-\frac{3}{4}y)^2 \Delta y$



$\frac{8}{y} = \frac{6}{x}$

$x = \frac{6}{8}y$

$$\therefore\ W = \int_0^8 \pi \rho g (6-\tfrac{3}{4}y)^2 (y+5)\,dy$$

mass of slice:

$\Delta m = \pi\rho(6-\frac{3}{4}y)^2\Delta y$

force of slice:

$\Delta F = \pi\rho g(6-\frac{3}{4}y)^2\Delta y$

work of slice:

$\Delta W = \Delta F d = \pi\rho g(6-\frac{3}{4}y)^2(y+5)\Delta y$

Question 3. *(5 marks)* Find the arc length of the graph of

$$y=\frac{x^3}{6}+\frac{1}{2x} \qquad y' = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{x^2}{2} - \frac{1}{2x^2}$$

on the interval $[1,2]$

$$S = \int_1^2 \sqrt{1+(y')^2}\,dx$$

$$= \int_1^2 \sqrt{1+\left(\frac{x^2}{2}-\frac{1}{2x^2}\right)^2}\,dx$$

$$= \int_1^2 \sqrt{1+\frac{x^4}{4}-\frac{1}{2}+\frac{1}{4x^4}}\,dx$$

$$= \int_1^2 \sqrt{\frac{x^4}{4}+\frac{1}{2}+\frac{1}{4x^4}}\,dx$$

$$= \int_1^2 \sqrt{\left(\frac{x^2}{2}+\frac{1}{2x^2}\right)^2}\,dx$$

$$= \int_1^2 \frac{x^2}{2}+\frac{1}{2x^2}\,dx$$

$$= \left[\frac{x^3}{6}-\frac{1}{2x}\right]_1^2$$

$$= \frac{2^3}{6}-\frac{1}{2(2)}-\frac{1}{6}+\frac{1}{2}$$

$$= \frac{17}{12}$$

Question 4. *(5 marks)* Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{n!\tan\left(\frac{1}{n}\right)}{(n-1)!} \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n!\tan\left(\frac{1}{n}\right)}{(n-1)!}$$

$$= \lim_{n\to\infty} \frac{n\cancel{(n-1)!}\tan\left(\frac{1}{n}\right)}{\cancel{(n-1)!}}$$

$$= \lim_{n\to\infty} n\tan\left(\frac{1}{n}\right) \qquad \text{I.F. } \infty\cdot 0$$

continuous over $\mathbb{R}$

$$= \lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{1/n} \qquad \text{I.F. } \frac{0}{0}$$

$$= \lim_{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right)\cancel{\frac{-1}{n^2}}}{\cancel{\frac{-1}{n^2}}} \qquad \text{by } \hat{H}$$

$$= 1$$

Question 5. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{1+\pi^{n-1}}{e^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{e^{n+1}} + \sum_{n=1}^{\infty} \frac{\pi^{n-1}}{e^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{ee^{n}} + \sum_{n=1}^{\infty} \frac{1}{\pi e}\left(\frac{\pi}{e}\right)^{n}$$

geometric series where $r = \frac{\pi}{e} > 1$

$\therefore$ diverges

Question 6. *(5 marks)* Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=2}^{\infty} [\arctan(n-1) - \arctan(n+1)]$$

$$S_n = a_2 + a_3 + a_4 + a_5 + a_6 + \dots + a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} + a_n$$

$$= [\arctan 1 - \cancel{\arctan 3}] + [\arctan 2 - \cancel{\arctan 4}] + [\cancel{\arctan 3} - \cancel{\arctan 5}]$$

$$+ [\cancel{\arctan 4} - \arctan 6] + [\cancel{\arctan 5} - \arctan 7] + \dots + [\arctan(n-5) - \cancel{\arctan(n-3)}]$$

$$+ [\arctan(n-4) - \cancel{\arctan(n-2)}] + [\cancel{\arctan(n-3)} - \cancel{\arctan(n-1)}]$$

$$+ [\cancel{\arctan(n-2)} - \arctan n] + [\cancel{\arctan(n-1)} - \arctan(n+1)]$$

$$= \arctan 1 + \arctan 2 - \arctan n - \arctan(n+1)$$

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[\arctan 1 + \arctan 2 - \underbrace{\arctan n}_{\pi/2} - \underbrace{\arctan(n+1)}_{\pi/2}\right]$$

$$= \frac{\pi}{4} + \arctan 2 - \pi$$

Question 7. *(5 marks)* If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = \operatorname{arcsec} n$$

find a_n and $\sum_{n=1}^{\infty}$.

$S_n = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1} + a_n$

$S_{n-1} = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1}$

$\therefore \quad S_n - S_{n-1} = a_n$

$a_n = \operatorname{arcsec} n - \operatorname{arcsec}(n-1) \quad \forall n \geq 2$

for $n = 1$ $\quad a_1 = S_1 = \operatorname{arcsec} 1 = 0$

$S = \lim_{n\to\infty} S_n$

$= \lim_{n\to\infty} \operatorname{arcsec} n$

$= \frac{\pi}{2}$

Question 8. Determine whether the following series converges or diverges. Justify your answer.

$$\sum_{n=3}^{\infty} \underbrace{\frac{\sqrt{n+1}}{n\sqrt{n^2-n-1}}}_{a_n}$$

Let $\sum_{n=3}^{\infty} b_n$ where $b_n = \frac{\sqrt{n}}{n\sqrt{n^2}} = \frac{1}{n^{3/2}}$. The series converges since p-series where $p = \frac{3}{2} > 1$.

Observe:

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\cancel{n}\sqrt{n^2-n-1}} \cdot \frac{n\sqrt{n^2}}{\sqrt{n}}$

$= \lim_{n\to\infty} \sqrt{\frac{n^2(n+1)}{n(n^2-n-1)}}$

$= \lim_{n\to\infty} \sqrt{\frac{n^3+n^2}{n^3-n^2-n}}$

$= 1$ finite and positive

$\therefore$ the series converges by limit comparison test.

Question 9. Determine whether the following series converges or diverges. Justify your answer.

$$\sum_{n=5}^{\infty} \frac{2^n n!}{n(2n)!}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)(2(n+1))!} \cdot \frac{n(2n)!}{2^n n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n} 2 \cancel{(n+1)} \cancel{n!}}{\cancel{(n+1)}(2n+2)!} \frac{n(2n)!}{\cancel{2^n} \cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{2n \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= 0 < 1$$

$\therefore$ by the ratio test the series converges

Question 10. Prove: If $p < 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

diverges.

Lets use the integral test and $f(x) = \frac{1}{x^p}$.

- $f(x)$ is positive for $x \geq 1$.
- $f(x)$ is continuous for $x \geq 1$.
- $f'(x) = \frac{-p}{x^{p-1}} < 0$ for $x \geq 1$. $\therefore$ $f(x)$ decreasing.

$$\int_1^{\infty} \frac{1}{x^p} = \lim_{b\to\infty} \int_1^b \frac{1}{x^p} dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{p+1} \right]_1^b$$

$$= \lim_{b\to\infty} \frac{\cancel{b^{-p+1}}^{\nearrow\infty}}{\cancel{p+1}} - \frac{1^{-p+1}}{p+1}$$

$\therefore$ diverges since $-p+1 > 0$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges by integral test.

Bonus Question.

a. *(1 mark)* State the $K(\varepsilon)$ definition of the limit of a sequence.

b. *(3 marks)* Using the $K(\varepsilon)$ definition show that

$$\lim_{n\to\infty} \frac{(-1)^n n}{n^2+1} = 0$$

c. *(3 marks)* Using the $K(\varepsilon)$ definition show: If a_n converges to a and b_n converges to b then $a_n + b_n$ converges to $a+b$.

a) $\forall \varepsilon > 0 \ \exists K(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$.

b) Given $\varepsilon > 0$, notice

$$\left| \frac{(-1)^n n}{n^2+1} - 0 \right| = \left| \frac{(-1)^n n}{n^2+1} \right|$$

$$= \frac{n}{n^2+1}$$

$$< \frac{n}{n^2} = \frac{1}{n}$$

Then by the Archeamedian property $\exists K(\varepsilon) \in \mathbb{N}$ such that

$$\frac{1}{\varepsilon} < K(\varepsilon)$$

$$\frac{1}{K(\varepsilon)} < \varepsilon$$

Thus $\forall \varepsilon > 0 \quad \exists K(\varepsilon) \in \mathbb{N}$ s.t. $\left| \frac{(-1)^n n}{n^2+1} \right| < \varepsilon$.

c) Given $\varepsilon > 0$, there exists $K_a(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2} \quad \forall n \geq K_a(\varepsilon)$

there exists $K_b(\varepsilon) \in \mathbb{N}$ s.t. $|b_n - b| < \frac{\varepsilon}{2} \quad \forall n \geq K_b(\varepsilon)$

let $K(\varepsilon) = \max\{K_a(\varepsilon), K_b(\varepsilon)\}$, so

$$|a_n + b_n - (a+b)| = |a_n - a + b_n - b|$$

$\leq |a_n - a| + |b_n - b|$ by the triangle inequality

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ by hypothesis and choice of $K(\varepsilon)$.

$n \geq K(\varepsilon)$

$= \varepsilon$

$\therefore \quad a_n + b_n \to a + b$ as $n \to \infty$.