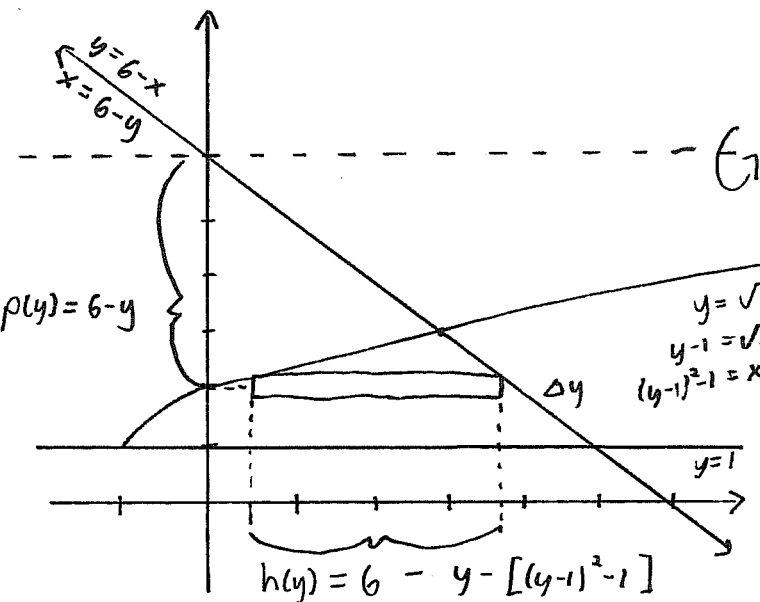


Test 3

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (5 marks) Set up the integral to find the volume of the solid obtained from the region bounded by the graphs of $y = \sqrt{x+1} + 1$, $y = 6-x$ and $y = 1$ rotated about the line $y = 6$.



Intersection of $y=6-x$, $y=\sqrt{x+1}+1$

$$6-x = \sqrt{x+1} + 1$$

$$5-x = \sqrt{x+1}$$

$$(5-x)^2 = x+1$$

$$25 - 10x + x^2 = x+1$$

$$0 = x^2 - 11x + 24$$

$$0 = (x-3)(x-8)$$

$$x=3$$

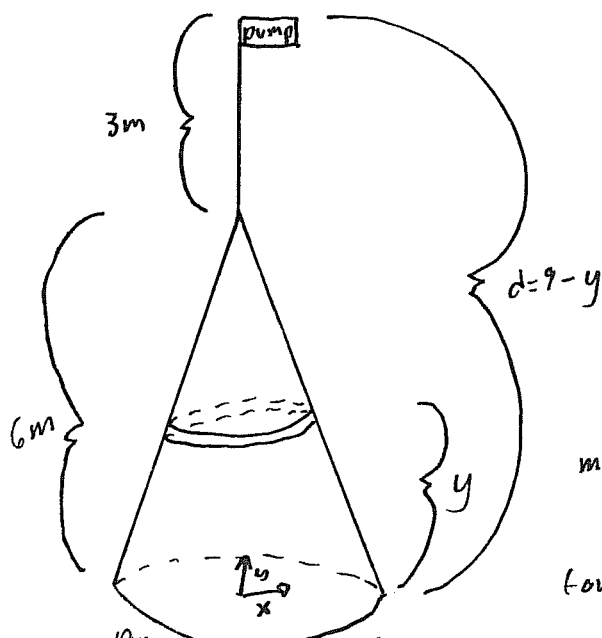
∴ volume of a shell

$$\Delta V = 2\pi p(y)h(y)\Delta y$$

$$= 2\pi(6-y)(6-y - [(y-1)^2 - 1])\Delta y$$

$$V = \int_1^3 2\pi(6-y)(6-y - [(y-1)^2 - 1]) dy$$

Question 2. (5 marks) A tank which has the shape of a cone with its tip upwards has a 10m diameter on the bottom and a height of 6m. Set up the integral to find the work required to empty the tank of with a liquid with an arbitrary density ρ from a pump 3m above the tip of the tank.



Volume of slice :

$$\Delta V = \pi r^2 \Delta y$$

$$= \pi (5-x)^2 \Delta y$$

$$= \pi (5 - \frac{5}{6}y)^2 \Delta y$$



$$\frac{6}{y} = \frac{5}{x}$$

$$x = \frac{5y}{6}$$

mass of slice :

$$\Delta m = \rho \Delta V = \rho \pi (5 - \frac{5}{6}y)^2 \Delta y$$

force of slice

$$\Delta F = \Delta m g = \rho g \pi (5 - \frac{5}{6}y)^2 \Delta y$$

work of slice :

$$\Delta W = \Delta F d$$

$$= \rho g \pi (5 - \frac{5}{6}y)^2 (9-y) \Delta y$$

$$\therefore W = \int_0^6 \rho g \pi (5 - \frac{5}{6}y)^2 (9-y) dy$$

Question 3. (5 marks) Find the arc length of the graph of

$$y = \frac{x^2}{2} - \frac{\ln x}{4} \quad y' = \frac{2x}{2} - \frac{1}{4x}$$

on the interval $[1, e]$

$$S = \int_1^e \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} dx$$

$$= \int_1^e \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}} dx$$

$$= \int_1^e \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}} dx$$

$$= \int_1^e \sqrt{\left(x + \frac{1}{4x}\right)^2} dx$$

$$= \int_1^e \left(x + \frac{1}{4x}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{\ln|x|}{4} \right]_1^e$$

$$= \frac{e^2}{2} + \frac{\ln e}{4} - \frac{1}{2}$$

$$= \frac{e^2}{2} - \frac{1}{4}$$

Question 4. (5 marks) Determine whether the sequence converges or diverges. If it converges, find the limit.

$$a_n = \frac{n! \sin\left(\frac{1}{n}\right)}{(n-1)!}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n! \sin\left(\frac{1}{n}\right)}{(n-1)!}$$

$$= \lim_{n \rightarrow \infty} \frac{n \cancel{(n-1)!} \sin\left(\frac{1}{n}\right)}{\cancel{(n-1)!}}$$

\checkmark continuous over \mathbb{R}

$$= \lim_{n \rightarrow \infty} n \sin\left(\frac{1}{n}\right) \quad \text{l.f. } \infty \cdot 0$$

$$= \lim_{n \rightarrow \infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \quad \text{l.f. } \frac{0}{0}$$

$$= \lim_{n \rightarrow \infty} \frac{\cos\left(\frac{1}{n}\right) \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}} \quad \text{by } \hat{H}$$

$$= 1$$

Question 5. (5 marks) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2 + \pi^{n+1}}{e^{n-1}} &= \sum_{n=1}^{\infty} \frac{2}{e^{n-1}} + \sum_{n=1}^{\infty} \frac{\pi^{n+1}}{e^{n-1}} \\ &= \sum_{n=1}^{\infty} \frac{2e}{e^n} + \sum_{n=1}^{\infty} \pi e \left(\frac{\pi}{e}\right)^n \end{aligned}$$

↑ geometric series where $r = \frac{\pi}{e} > 1$

∴ diverges

Question 6. (5 marks) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=2}^{\infty} [\operatorname{arcsec}(n-1) - \operatorname{arcsec}(n+1)]$$

$$S_n = a_2 + a_3 + a_4 + a_5 + \dots + a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} + a_n$$

$$= [\operatorname{arcsec} 1 - \operatorname{arcsec} 3] + [\operatorname{arcsec} 2 - \operatorname{arcsec} 4] + [\operatorname{arcsec} 3 - \operatorname{arcsec} 5]$$

$$+ [\operatorname{arcsec} 4 - \operatorname{arcsec} 6] + [\operatorname{arcsec} 5 - \operatorname{arcsec} 7] + \dots + [\operatorname{arcsec}(n-5) - \operatorname{arcsec}(n-3)]$$

$$+ [\operatorname{arcsec}(n-4) - \operatorname{arcsec}(n-2)] + [\operatorname{arcsec}(n-3) - \operatorname{arcsec}(n-1)] + [\operatorname{arcsec}(n-2) - \operatorname{arcsec} n]$$

$$+ [\operatorname{arcsec}(n-1) - \operatorname{arcsec}(n+1)]$$

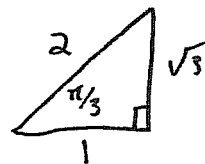
$$= \operatorname{arcsec} 1 + \operatorname{arcsec} 2 - \operatorname{arcsec} n - \operatorname{arcsec}(n+1)$$

$$= 0 + \frac{\pi}{3} - \operatorname{arcsec} n - \operatorname{arcsec}(n+1)$$

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[\frac{\pi}{3} - \operatorname{arcsec} n - \operatorname{arcsec}(n+1) \right]$$

$$= \frac{\pi}{3} - \frac{\pi}{2} - \frac{\pi}{2}$$

$$= -\frac{2\pi}{3}$$



Question 7. (5 marks) If the n^{th} partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$S_n = \arctan n$$

$$S_n = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1} + a_n$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

$$S_{n-1} = a_1 + a_2 + a_3 + \dots + a_{n-2} + a_{n-1}$$

$$\therefore S_n - S_{n-1} = a_n$$

$$a_n = \arctan n - \arctan(n-1)$$

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \arctan n = \frac{\pi}{2}$$

Question 8. (5 marks) Determine whether the following series converges or diverges. Justify your answer.

$$\sum_{n=3}^{\infty} \underbrace{\frac{\sqrt{n+1}}{n\sqrt{n-\sqrt{n}-1}}}_{a_n}$$

Let $\sum_{n=3}^{\infty} b_n$ where $b_n = \frac{\sqrt{n}}{n\sqrt{n}} = \frac{1}{n}$. The

series diverges since p -series where $p=1$.

Observe:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n+1}}{n\sqrt{n-\sqrt{n}-1}} \cdot \frac{n\sqrt{n}}{\sqrt{n}}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{n(n+1)}{n(n-\sqrt{n}-1)}}$$

$$= \lim_{n \rightarrow \infty} \sqrt{\frac{n^2+n}{n^2-n^{3/2}-n}}$$

$$= 1 \text{ finite and positive}$$

\therefore the series diverges by the limit comparison test.

Question 9. (5 marks) Determine whether the following series converges or diverges. Justify your answer.

$$\sum_{n=5}^{\infty} \frac{n(2n)!}{3^n n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

\therefore by ratio test diverges.

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(2(n+1))!}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n(2n)!} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{(n+1)(2n+2)!}{3^n 3(n+1)n!} \cdot \frac{3^n n!}{n(2n)!} \right|$$

$$= \lim_{n \rightarrow \infty} \frac{(2n+2)(2n+1)(2n)!}{3n(2n)!} > 1$$

Question 10. (5 marks) Prove: If $p > 1$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges.

Let's use the integral test and $f(x) = \frac{1}{x^p}$

- $f(x)$ is positive for $x \geq 1$.
- $f(x)$ is continuous for $x \geq 1$.
- $f'(x) = \frac{-p}{x^{p+1}} < 0$ for $x \geq 1$ \therefore $f(x)$ decreasing

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^p} dx \\ &= \lim_{b \rightarrow \infty} \left[\frac{x^{-p+1}}{-p+1} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \\ &= \frac{-1}{p-1} \end{aligned}$$

since $-p+1 < 0$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges}$$

\therefore converges

by integral test.

Bonus Question.

a. (1 mark) State the $K(\varepsilon)$ definition of the limit of a sequence.

b. (3 marks) Using the $K(\varepsilon)$ definition show that

$$\lim_{n \rightarrow \infty} \frac{(-1)^n n}{n^2 + 1} = 0$$

c. (3 marks) Using the $K(\varepsilon)$ definition show: If a_n converges to a and b_n converges to b then $a_n + b_n$ converges to $a + b$.

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a) $\forall \varepsilon > 0 \exists K(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon \quad \forall n \geq K(\varepsilon)$.

b) Given $\varepsilon > 0$, notice

$$\begin{aligned} \left| \frac{(-1)^n n}{n^2 + 1} - 0 \right| &= \left| \frac{(-1)^n n}{n^2 + 1} \right| \\ &= \frac{n}{n^2 + 1} \\ &< \frac{n}{n^2} = \frac{1}{n} \end{aligned}$$

Then by the Archimedean property $\exists K(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} \frac{1}{\varepsilon} &< K(\varepsilon) \\ \frac{1}{K(\varepsilon)} &< \varepsilon \end{aligned}$$

Thus $\forall \varepsilon > 0 \exists K(\varepsilon) \in \mathbb{N}$ s.t. $\left| \frac{(-1)^n n}{n^2 + 1} \right| < \varepsilon$.

c) Given $\varepsilon > 0$, there exists $K_a(\varepsilon) \in \mathbb{N}$ s.t. $|a_n - a| < \frac{\varepsilon}{2} \quad \forall n \geq K_a(\varepsilon)$
there exists $K_b(\varepsilon) \in \mathbb{N}$ s.t. $|b_n - b| < \frac{\varepsilon}{2} \quad \forall n \geq K_b(\varepsilon)$

let $K = \max\{K_a(\varepsilon), K_b(\varepsilon)\}$, so

$$\begin{aligned} |a_n + b_n - (a + b)| &= |a_n - a + b_n - b| \\ &\leq |a_n - a| + |b_n - b| \text{ by the triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \text{ by hypothesis and choice of } K(\varepsilon). \\ &= \varepsilon \text{ } \swarrow n \geq K(\varepsilon) \end{aligned}$$

$\therefore a_n + b_n \rightarrow a + b$ as $n \rightarrow \infty$.