Dawson College: **Linear Algebra**: 201-NYC-05-S07: Winter 2010

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Test 1

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(10 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcl} 3x_1 & + & 2x_2 & + & 2x_3 & & & = & 1 \\ 4x_1 & + & 1x_2 & - & 3x_3 & + & x_4 & = & 2 \\ 7x_1 & + & 3x_2 & - & 1x_3 & + & x_4 & = & 3 \end{array}$$

In addition, if the solution is not unique give two particular solutions.

$$\left[\begin{array}{ccccc} 3 & 2 & 2 & 0 & 1 \\ 4 & 1 & -3 & 1 & 2 \\ 7 & 3 & -1 & 1 & 3 \end{array}\right]$$

$$\sim \begin{array}{r} \\ 3R_2 \\ 3R_3 \end{array} \left[\begin{array}{ccccc} 3 & 2 & 2 & 0 & 1 \\ 12 & 3 & -9 & 3 & 6 \\ 21 & 9 & -3 & 3 & 9 \end{array}\right]$$

$$\sim \begin{array}{r} \\ -4R_1+R_2 \to R_2 \\ -7R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccccc} 3 & 2 & 2 & 0 & 1 \\ 0 & -5 & -17 & 3 & 2 \\ 0 & -5 & -17 & 3 & 2 \end{array}\right]$$

$$\sim \begin{array}{r} 5R_1 \\ \\ -R_2+R_3 \to R_3 \end{array} \left[\begin{array}{ccccc} 15 & 10 & 10 & 0 & 5 \\ 0 & -5 & -17 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{r} 2R_2+R_1 \to R_1 \\ \\ \\ \end{array} \left[\begin{array}{ccccc} 15 & 0 & -24 & 6 & 9 \\ 0 & -5 & -17 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{r} \frac{1}{15}R_1 \\ -\frac{1}{5}R_2 \\ \\ \end{array} \left[\begin{array}{ccccc} 1 & 0 & -\frac{24}{15} & 6/15 & 9/15 \\ 0 & 1 & \frac{17}{5} & -\frac{3}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variables x_3, x_4.

So $x_3 = s$, $x_4 = t$

$x_1 = \frac{24}{15}s - \frac{6}{15}t + \frac{9}{15}$

$x_2 = -\frac{17}{5}s + \frac{3}{5}t - \frac{2}{5}$

$x_3 = s$

$x_4 = t$

not unique solution.

Let $s = 0 = t$

$\Rightarrow x_1 = \frac{9}{15}$

$x_2 = -\frac{2}{5}$

$x_3 = 0$

$x_4 = 0$

Let $s = 0,\ t = 1$

$\Rightarrow x_1 = \frac{-6}{15}(1) + \frac{9}{15} = \frac{3}{15}$

$x_2 = +\frac{3}{5}(1) - \frac{2}{5} = \frac{1}{5}$

$x_3 = 0$

$x_4 = 1$

Question 2. *(5 marks)* Consider the matrices:

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & -1 \\ 1 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Then compute $\mathrm{tr}(AA^t - 2B)$, if possible.

$$AA^t - 2B = \begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & -1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 1 \\ 2 & 3 & 0 \\ 1 & -1 & -1 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 & -1 \\ 0 & -4 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 5 & -2 \\ 5 & 10 & 1 \\ -2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 4 & -2 \\ 0 & -8 & 2 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 5 & 18 & -1 \\ -4 & -3 & 0 \end{bmatrix}$$

$\therefore \mathrm{tr}(AA^t - 2B) = 18$

Question 3. *(5 marks)* Solve for the matrix X if

$$X\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 6 \end{bmatrix}$$

2×2 $\quad 2\times 3 \quad$ 2×3

$\therefore X$ is of dim. 2×2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} a & a & a+2b \\ c & c & c+2d \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 4 & 4 & 6 \end{bmatrix}$$

$\therefore$ $a = 1$ (1) $\quad a + 2b = 3$ (3)

$c = 4$ (2) $\quad c + 2d = 6$ (4)

sub (1) into (3) $\quad 1 + 2b = 3$

$2b = 2$

$b = 1$

sub (2) into (4) $\quad 4 + 2d = 6$

$2d = 2$

$d = 1$

$$\therefore X = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

Question 4. *(10 marks)* Solve the following system using the inverse of the coefficient matrix.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & -1 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 2x_2 & + & 4x_3 & = & 1 \end{array}$$

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 4 \\ 1 & -2 & 4 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$Ax = b$

Lets find the inverse of A.

$[A \mid I]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & +3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 0 & 1 & 0 \\ 1 & +2 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -R_1 + R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & +3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -3R_3 + R_1 \to R_1 \\ 2R_3 + R_2 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 4 & 0 & -3 \\ 0 & -1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim 2R_2 + R_1 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & -1 & 0 & -4 & 1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim -R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 4 & -1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore x = A^{-1}b = \begin{bmatrix} -4 & 2 & 1 \\ 4 & -1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ 2 \end{bmatrix}$$

Question 5. Express

$$A=\begin{bmatrix}3 & 4\\5 & -2\end{bmatrix} B=\begin{bmatrix}3 & 4\\11 & 6\end{bmatrix}, C=\begin{bmatrix}11 & 6\\3 & 4\end{bmatrix}$$

Find the elementary matrices E_1 and E_2 (if possible) such that

a. *(2 marks)* $E_1A = B$

b. *(2 marks)* $E_2A = C$

a) $A = \begin{bmatrix}3 & 4\\5 & -2\end{bmatrix} \sim_{2R_1+R_2 \to R_2} \begin{bmatrix}3 & 4\\11 & 6\end{bmatrix} = B$ $\therefore$ $I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \sim_{2R_1+R_2 \to R_2} \underbrace{\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}}_{= E_1}$

b) $B = \begin{bmatrix}3 & 4\\11 & 6\end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix}11 & 6\\3 & 4\end{bmatrix} = C$

$\therefore$ two elementary row operation to get from A to C

$\therefore$ E_2 does not exists.

Question 6. *(3 marks)* Show that if A, D are invertible and $ABD = ACD$ then $B = C$.

Given $AA^{-1} = I = A^{-1}A$ so

$DD^{-1} = I = D^{-1}D$

$$ABD = ACD$$
$$A^{-1}ABDD^{-1} = A^{-1}ACDD^{-1}$$
$$IBI = ICI$$
$$B = C$$

Question 7. *(3 marks)* Find A^{-1} if $A^3 + 3A - I = 0$.

$$A^3 + 3A = I$$

$$A(A^2 + 3I) = I \qquad (A^2 + 3I)A = I$$

$\therefore$ $A^{-1} = A^2 + 3I$

Question 8. *(5 marks)* If D is an $n \times n$ diagonal matrix, A is an $n \times n$ symmetric matrix and B is an $n \times n$ matrix then show that $B^tB + 2A + D - I$ is symmetric.

We must show $(B^tB + 2A + D - I)^t = B^tB + 2A + D - I$

$$\begin{aligned}
LHS &= (B^tB + 2A + D - I)^t \\
&= (B^tB)^t + (2A)^t + D^t - I^t \\
&= B^t(B^t)^t + 2A^t + D^t - I^t \\
&= B^tB + 2A + D - I \\
&= RHS
\end{aligned}$$

note: $A^t = A$, $D^t = D$, $I^t = I$

$\therefore$ symmetric.

Question 9. *(5 marks)* For which value(s) of the constant a and b does the system

$$\begin{aligned} x &+ y &= 1 \\ 4x &+ a^2 y &= b \end{aligned}$$

have no solutions? Exactly one solution? Infinitely many solutions? Justify.

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & a^2 & b \end{bmatrix} \sim_{-4R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & a^2-4 & b-4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & (a-2)(a+2) & b-4 \end{bmatrix}$$

if $a=-2,2$ and $b\neq 4$ then the system is inconsistent $\therefore$ no solution

If $a\neq -2,2$ then # leading 1 = # variables $\therefore$ unique solution

If $a=-2,2$ and $b=4$ then there is a free variable $\therefore$ Infinitely many solutions.

Bonus Question. *(3 marks)* Prove: If $A^5=0$ then

$$(I-A)^{-1} = I+A+A^2+A^3+A^4.$$

$$\begin{aligned}(I-A)(I-A)^{-1} &= (I-A)(I+A+A^2+A^3+A^4) \\ &= I+\cancel{A}+\cancel{A^2}+\cancel{A^3}+\cancel{A^4}-\cancel{A}-\cancel{A^2}-\cancel{A^3}-\cancel{A^4}-A^5 \\ &= I-A^5 \\ &= I \quad \text{since } A^5=0\end{aligned}$$

$$\begin{aligned}(I-A)^{-1}(I-A) &= (I+A+A^2+A^3+A^4)(I-A) \\ &= I+\cancel{A}+\cancel{A^2}+\cancel{A^3}+\cancel{A^4}-\cancel{A}-\cancel{A^2}-\cancel{A^3}-\cancel{A^4}-A^5 \\ &= I-A^5 \\ &= I \quad \text{since } A^5=0\end{aligned}$$