Dawson College: **Linear Algebra**: 201-NYC-05-S07: Winter 2010

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Test 1

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(10 marks)* Solve the following system by Gauss-Jordan elimination:

$$\begin{array}{rcrcrcrcl} 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & = & 1 \\ 3x_1 & + & 2x_2 & - & 2x_3 & + & & = & 2 \\ 5x_1 & + & 5x_2 & + & x_3 & + & x_4 & = & 3 \end{array}$$

In addition, if the solution is not unique give two particular solutions.

$$\begin{bmatrix} 2 & 3 & 3 & 1 & 1 \\ 3 & 2 & -2 & 0 & 2 \\ 5 & 5 & 1 & 1 & 3 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ 2R_2 \\ 2R_3 \end{array} \begin{bmatrix} 2 & 3 & 3 & 1 & 1 \\ 6 & 4 & -4 & 0 & 4 \\ 10 & 10 & 2 & 2 & 6 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ -3R_1 + R_2 \to R_2 \\ -5R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 3 & 3 & 1 & 1 \\ 0 & -5 & -13 & -3 & 1 \\ 0 & -5 & -13 & -3 & 1 \end{bmatrix}$$

$$\sim \begin{array}{r} \\ \\ -R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 3 & 3 & 1 & 1 \\ 0 & -5 & -13 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} 5R_1 \\ \\ \\ \end{array} \begin{bmatrix} 10 & 15 & 15 & 5 & 5 \\ 0 & -5 & -13 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} 3R_2 + R_1 \to R_1 \\ \\ \\ \end{array} \begin{bmatrix} 10 & 0 & -24 & -4 & 8 \\ 0 & -5 & -13 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{r} \frac{1}{10}R_1 \\ -\frac{1}{5}R_2 \\ \\ \end{array} \begin{bmatrix} 1 & 0 & -\frac{12}{5} & -\frac{2}{5} & 4/5 \\ 0 & 1 & 13/5 & 3/5 & -1/5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

free variables X_3, X_4

$\therefore X_3 = s, \; X_4 = t$

$\therefore X_1 = \frac{4}{5} + \frac{2}{5}s - \frac{12}{5}t$

$X_2 = \frac{-1}{5} - \frac{13}{5}s - \frac{3}{5}t$

$X_3 = s$

$X_4 = t$

not a unique solution.

Let $s = t = 0$

$\therefore (x_1, x_2, x_3, x_4) = (4/5, -1/5, 0, 0)$

is a particular solution

Let $s = t = 1$

$\therefore (x_1, x_2, x_3, x_4) = (\frac{-6}{5}, \frac{-17}{5}, 1, 1)$

Question 2. *(5 marks)* Consider the matrices:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}, B = \begin{bmatrix} 3 & 3 & 1 \\ 0 & -4 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

Then compute $\operatorname{tr}(AA^t - 3B)$, if possible.

$$AA^t - 3B = \begin{bmatrix} 0 & 2 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} - 3\begin{bmatrix} 3 & 3 & 1 \\ 0 & -4 & 1 \\ 1 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 9 & 3 \\ 0 & -12 & 3 \\ 3 & -6 & -3 \end{bmatrix} = \begin{bmatrix} -4 & -10 & -2 \\ -1 & 14 & -3 \\ -2 & 6 & 6 \end{bmatrix}$$

$$\therefore \operatorname{tr}(AA^t - 3B) = -4 + 14 + 6 = 16$$

Question 3. *(5 marks)* Solve for the matrix X if

$$X\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 4 & 5 \end{bmatrix}$$

$2\times2 \quad 2\times3 \qquad 2\times3$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 4 & 5 \end{bmatrix}$

$$\begin{bmatrix} a & a+2b & a \\ c & c+2d & c \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 5 & 4 & 5 \end{bmatrix}$$

$\therefore$ (1) $a = 2$ (3) $a + 2b = 3$

(2) $c = 5$ (4) $c + 2d = 4$

sub (1) into (3): $a + 2b = 3$, $2b = 1$, $b = 1/2$

sub (2) into (4): $5 + 2d = 4$, $2d = -1$, $d = \frac{-1}{2}$

$$\therefore X = \begin{bmatrix} 2 & 1/2 \\ 5 & \frac{-1}{2} \end{bmatrix}$$

Question 4. *(10 marks)* Solve the following system using the inverse of the coefficient matrix.

$$\begin{aligned} x_1 &+ 2x_2 + 2x_3 = 0 \\ x_1 &+ 3x_2 + 3x_3 = -1 \\ x_1 &+ 2x_2 + 3x_3 = 1 \end{aligned}$$

where $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$Ax = b$

Lets find the inverse of A.

$[A \mid I]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 3 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \begin{array}{l} -2R_3+R_1 \to R_1 \\ -R_3+R_2 \to R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim -2R_2+R_1 \to R_1 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

So $x = A^{-1}b$

$$= \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Question 5. Express

$$A = \begin{bmatrix} -3 & 5 \\ 6 & -2 \end{bmatrix} B = \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix}, C = \begin{bmatrix} 0 & 8 \\ -3 & 5 \end{bmatrix}$$

Find the elementary matrices E_1 and E_2 (if possible) such that

a. *(2 marks)* $E_1A = B$

b. *(2 marks)* $E_2A = C$

a) notice: $A = \begin{bmatrix} -3 & 5 \\ 6 & -2 \end{bmatrix} \sim_{2R_1 + R_2 \to R_2} \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix} = B$

So $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim_{2R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_1$

b) notice: $B = \begin{bmatrix} -3 & 5 \\ 0 & 8 \end{bmatrix} \sim_{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 8 \\ -3 & 5 \end{bmatrix} = C$

$\therefore$ no elementary matrices to get from A to C since it requires two elementary row operations.

Question 6. *(3 marks)* Show that if A, D are invertible and $ABD = 0$ then $B = 0$.

From hypothesis we have $DD^{-1} = I = D^{-1}D$ and $AA^{-1} = I = A^{-1}A$

$$\begin{aligned} ABD &= 0 \\ A^{-1}ABDD^{-1} &= A^{-1}0D^{-1} \\ IBI &= 0 \\ B &= 0 \end{aligned}$$

Question 7. *(3 marks)* Find A^{-1} if $A^4 + 2A - I = 0$.

$$A^4 + 2A = I$$

$$A\underbrace{(A^3 + 2I)}_{A^{-1}} = I \qquad \underbrace{(A^3 + 2I)}_{A^{-1}}A = I$$

$\therefore A^{-1} = A + 2I$

Question 8. *(5 marks)* If D is an $n \times n$ diagonal matrix, A is an $n \times n$ symmetric matrix and B is an $n \times n$ matrix then show that $B^tB + 2A + D - I$ is symmetric.

We must show $(B^tB + 2A + D - I)^t = B^tB + 2A + D - I$

$$\begin{aligned} LHS &= (B^tB + 2A + D - I)^t \\ &= (B^tB)^t + (2A)^t + D^t - I^t \\ &= B^t(B^t)^t + 2A^t + D^t - I^t \\ &= B^tB + 2A + D - I \\ &= RHS \end{aligned}$$

note: $A^t = A$, $D^t = D$, $I^t = I$

$\therefore$ symmetric.

Question 9. *(5 marks)* For which value(s) of the constant a and b does the system

$$\begin{array}{rcrcl} x & + & y & = & 1 \\ 9x & + & a^2 y & = & b \end{array}$$

have no solutions? Exactly one solution? Infinitely many solutions? Justify.

$$\begin{bmatrix} 1 & 1 & 1 \\ 9 & a^2 & b \end{bmatrix} \sim_{-9R_1+R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & a^2-9 & b-9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & (a-3)(a+3) & b-9 \end{bmatrix}$$

If $a = 3, -3$ and $b \neq 9$ then the system is inconsistent $\therefore$ no solution

If $a \neq 3, -3$ then # leading 1 = # variables $\therefore$ unique solution

If $a = 3, -3$ and $b = 9$ then there is a free variable $\therefore$ Infinitely many solutions.

Bonus Question. *(3 marks)* Prove: If $A^5 = 0$ then

$(I-A)^{-1} = I + A + A^2 + A^3 + A^4$.

$$\begin{aligned}(I-A)(I-A)^{-1} &= (I-A)(I+A+A^2+A^3+A^4) \\ &= I + \cancel{A} + \cancel{A^2} + \cancel{A^3} + \cancel{A^4} - \cancel{A} - \cancel{A^2} - \cancel{A^3} - \cancel{A^4} - A^5 \\ &= I - A^5 \\ &= I \quad \text{since } A^5 = 0\end{aligned}$$

$$\begin{aligned}(I-A)^{-1}(I-A) &= (I+A+A^2+A^3+A^4)(I-A) \\ &= I + \cancel{A} + \cancel{A^2} + \cancel{A^3} + \cancel{A^4} - \cancel{A} - \cancel{A^2} - \cancel{A^3} - \cancel{A^4} - A^5 \\ &= I - A^5 \\ &= I \quad \text{since } A^5 = 0\end{aligned}$$