Dawson College: **Linear Algebra**: 201-NYC-05-S07: Winter 2010

Name: YANN LAMONTAGNE
Student ID: ______

Test 2

This test is graded out of ~~50~~ 48 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. *(5 marks)* Use Cramer's rule to solve for x_2 without solving for x_1, x_3.

$$\begin{array}{rcrcrcl} 3x_1 & + & 2x_2 & + & 2x_3 & = & 1 \\ 4x_1 & + & x_2 & & & = & 0 \\ 7x_1 & + & 3x_2 & - & x_3 & = & 3 \end{array} \iff Ax=b \text{ where } A=\begin{bmatrix} 3 & 2 & 2 \\ 4 & 1 & 0 \\ 7 & 3 & -1 \end{bmatrix},\ b=\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

(Use cofactor expansions to find the determinants)

$$\det A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 4C_{21} + C_{22} + 0C_{23}$$

$$= 4(-1)^{2+1}\begin{vmatrix} 2 & 2 \\ 3 & -1 \end{vmatrix} + (-1)^{2+2}\begin{vmatrix} 3 & 2 \\ 7 & -1 \end{vmatrix}$$

$$= -4[-2-6] + [-3-14] = 15$$

$$A_2 = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 0 & 0 \\ 7 & 3 & -1 \end{bmatrix}$$

$$\det A_2 = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = 4C_{21} + 0C_{22} + 0C_{23}$$

$$= 4(-1)^{2+1}\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -4[-1-6] = 28$$

$$\therefore\ x_2 = \frac{\det A_2}{\det A} = \frac{28}{15}$$

Question 2. *(4 marks)* Show that if A, B are invertible then $\det(A^{-1}BCAB^{-1}) = \det(C)$

$$LHS = \det A^{-1} \det B \det C \det A \det B^{-1}$$

$$= \frac{1}{\cancel{\det A}} \cancel{\det B} \det C \cancel{\det A} \frac{1}{\cancel{\det B}} \quad \text{since } A \text{ and } B \text{ are invertible}$$

$$= \det C.$$

Question 3. *(5 marks)*

a. *(5 marks)* Use the combinatorial definition of the determinant to compute:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

b. *(2 marks)* Justify the "visual way" of computing the determinant using part a.

a) Let $S = \{1, 2\}$

permutation of S	# of inversions	parity of permutation	elementary product	Signed elementary product
$(1,2)$	0	even	$a_{11} a_{22}$	$+a_{11} a_{22}$
$(2,1)$	1	odd	$a_{12} a_{21}$	$-a_{12} a_{21}$

$\therefore \det A = a_{11} a_{22} - a_{12} a_{21}$

b)

$$|A| = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

Question 4.

a. *(5 marks)* Find the inverse of the following matrix using the adjoint:

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$$

b. *(2 marks)* Using part a. solve the following equation:

$$Ax = b$$

where

$$b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

a) $A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^t = \frac{1}{6} \begin{bmatrix} 0 & -3 \\ 2 & 1 \end{bmatrix}^t$

$\det A = +6$

$C_{11} = (-1)^{1+1} |0| = 0$

$C_{12} = (-1)^{1+2} |3| = -3$

$C_{21} = (-1)^{2+1} |-2| = 2$

$C_{22} = (-1)^{2+2} |1| = 1$

$\therefore A^{-1} = \frac{1}{6} \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}$

b)

$\therefore Ax = b$

$A^{-1} A x = A^{-1} b$

$x = A^{-1} b$

$$x = \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{2} & 1/6 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Question 5. *(3 marks)* For which value(s) of α is the following matrix invertible:

$$\begin{bmatrix} \alpha & -2 & 2 \\ 0 & \alpha-1 & 101 \\ 0 & 0 & \alpha^2-1 \end{bmatrix}$$

a matrix is invertible $\det \neq 0$

$$\begin{vmatrix} \alpha & -2 & 2 \\ 0 & \alpha-1 & 101 \\ 0 & 0 & \alpha^2-1 \end{vmatrix} = \alpha(\alpha-1)(\alpha^2-1) = \alpha(\alpha-1)^2(\alpha+1)$$

$\therefore$ if $\alpha \neq 0, -1, 1$ then the matrix is invertible.

Question 6. Let

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 0 & 0 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 0 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, D = \begin{bmatrix} b & 3a-2b \\ d & 3c-2d \end{bmatrix}$$

a) Notice $\det A = 0$

$\therefore \det AB$

$= \det A \det B$

$= 0$

$\therefore$ not invertible

a. *(3 marks)* If B is a 10×10 matrix show that AB is not invertible.

b. *(4 marks)* If $\det(D) = 2$ then find $\det(C)$.

$$C \sim \overset{C_1 \leftrightarrow C_2}{\begin{bmatrix} b & a \\ d & c \end{bmatrix}} \sim \overset{3C_2}{\begin{bmatrix} b & 3a \\ d & 3c \end{bmatrix}}$$

$$\sim \overset{-2C_1+C_2 \to C_2}{\begin{bmatrix} b & 3a-2b \\ d & 3c-2d \end{bmatrix}} = D$$

$(-1)(3) \det C = \det D$

$\det C = \frac{-2}{3}$

(5 marks)

Question 7. Compute the determinant of the following matrix using elementary operations

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ 4 & 3 & 1 \end{bmatrix}. \sim \begin{array}{c} \\ 2R_2 \\ \\ \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 6 & -4 & 2 \\ 4 & 3 & 1 \end{bmatrix} \sim \begin{array}{c} \\ -3R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -7 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ 7R_3 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -7 & -1 \\ 0 & 7 & -7 \end{bmatrix}$$

$$\sim \begin{array}{c} \\ \\ R_2+R_3 \to R_3 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -7 & -1 \\ 0 & 0 & -8 \end{bmatrix} = B$$

$$\det A = \det B$$

$$(2)(7) \det A = 2(-7)(-8)$$

$$\det A = 8$$

Question 8. *(5 marks)* If A, B are 3×3 matrices, $\det(2A) = -8$ and $\det(B) = \sqrt{2}$ then find

$\det((3AB)^{-1}(2AB)^t A^3 B^4 B^{-1})$.

(show every step)

notice:

$\det(2A) = -8$

$2^3 \det A = -8$

$\det A = (-1)$

$$\det((3AB)^{-1}(2AB)^t A^3 B^4 B^{-1})$$

$$= \det(3AB)^{-1} \det(2AB)^t \det A^3 \det B^4 \det B^{-1}$$

$$= \frac{1}{\det 3AB} \det 2AB \, (\det A)^3 (\det B)^{\cancel{4}\,3} \frac{1}{\cancel{\det B}}$$

$$= \frac{1}{3^3 \cancel{\det A} \cancel{\det B}} 2^3 \cancel{\det A} \cancel{\det B} (\det A)^3 (\det B)^3$$

$$= \frac{8}{27} (-1)^3 (\sqrt{2})^3$$

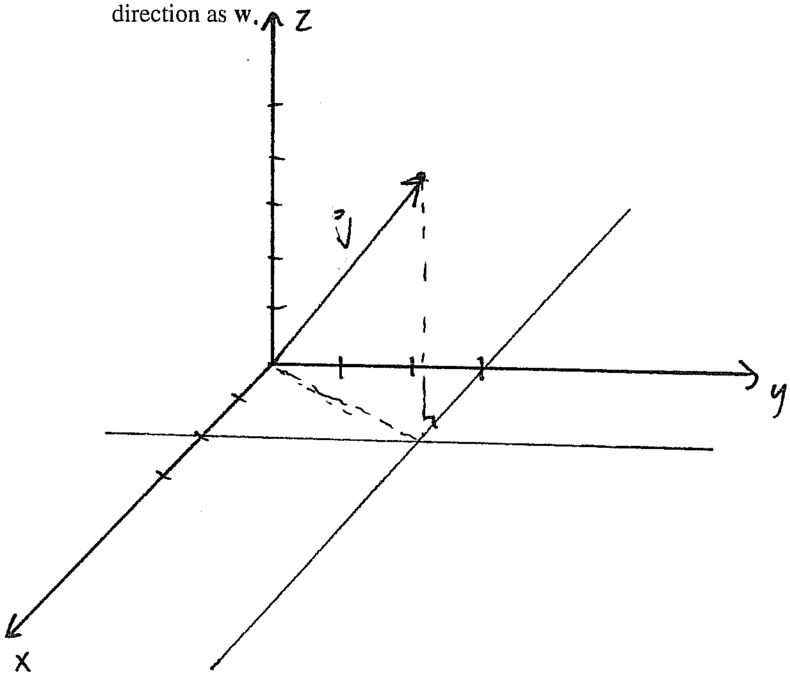
$$= \frac{-16\sqrt{2}}{27}$$

Question 9.

a. *(1 mark)* Sketch the vector $\mathbf{v} = (2,3,5)$ on a right-handed coordinate system.

b. *(2 marks)* Find a nonzero vector $\mathbf{u}$ with terminal point $Q(1,2,3)$ which is oppositely directed to $\mathbf{v}$.

c. *(2 marks)* Let $\mathbf{a} = (1,0,-2)$, $\mathbf{b} = (0,2,0)$ and $\mathbf{c} = (0,1,1)$. If $\mathbf{w} = 2(\mathbf{a}-\mathbf{b}) + 3\mathbf{c}$ then find a unit vector which has the same direction as $\mathbf{w}$.



b) Let $\vec{u}$ and $\vec{v}$ have the same magnitude but different direction

$\vec{u} = -\vec{v}$

$\vec{PQ} = -(2,3,5)$

$Q - P =$

$P = Q + (2,3,5)$

$P = (1,2,3) + (2,3,5)$

$P = (3,5,8)$

c) $\vec{w} = 2(\vec{a} - \vec{b}) + 3\vec{c}$

$= 2(1,0,-2) - 2(0,2,0) + 3(0,1,1)$

$= (2,0,-4) - (0,4,0) + (0,3,3)$

$= (2,-1,-1)$

$\therefore$ a unit vector that has the same direction

is $\dfrac{\vec{w}}{\|\vec{w}\|} = \dfrac{(2,-1,-1)}{\sqrt{2^2+(-1)^2+(-1)^2}} = \dfrac{(2,-1,-1)}{\sqrt{6}}$

Bonus Question. *(3 marks)* Consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

and use row reduction to show that $\det(A) = (b-a)(c-a)(c-b)$.

$$\begin{array}{l} \sim -aR_1 + R_2 \to R_2 \\ -a^2R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & (b-a)(b+a) & (c-a)(c+a) \end{bmatrix}$$

$$\sim \begin{array}{l} \\ \\ -(b+a)R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (a-c)(b+a) + (c-a)(c+a) \end{bmatrix} = B$$

$$\det A = \det B$$

$$\det A = 1 \cdot (b-a)\left[(a-c)(b+a) + (c-a)(c+a)\right]$$

$$\det A = (b-a)\left[-(c-a)(b+a) + (c-a)(c+a)\right]$$

$$\det A = (b-a)(c-a)\left[-(b+a) + c + a\right]$$

$$\det A = (b-a)(c-a)\left[c-b\right]$$