

Test 2

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. (5 marks) Use Cramer's rule to solve for x_2 without solving for x_1, x_3 .

$$\begin{aligned} 7x_1 + 2x_2 - 2x_3 &= 3 \\ 4x_1 + x_2 &= 0 \\ 3x_1 + 3x_2 - x_3 &= 1 \end{aligned} \Leftrightarrow Ax=b \text{ where } A = \begin{bmatrix} 7 & 2 & -2 \\ 4 & 1 & 0 \\ 3 & 3 & -1 \end{bmatrix} \text{ where } b = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

(Use cofactor expansions to find the determinants)

$$\begin{aligned} \det A &= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} = 4 C_{21} + C_{22} + 0 C_{23} \\ &= 4(-1)^{2+1} \begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} + (-1)^{2+2} \begin{vmatrix} 7 & -2 \\ 3 & -1 \end{vmatrix} \\ &= -4 [2(-1) - (-2)(3)] + [7(-1) - (-2)(3)] \\ &= -4 [4] + [-1] = -17 \end{aligned}$$

$$A_2 = \begin{bmatrix} 7 & 3 & -2 \\ 4 & 0 & 0 \\ 3 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} \det A_2 &= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} = 4 C_{21} + 0 C_{22} + 0 C_{23} \\ &= 4(-1)^{2+1} \begin{vmatrix} 3 & -2 \\ 1 & -1 \end{vmatrix} \end{aligned}$$

$$\therefore x_2 = \frac{\det A_2}{\det A} = \frac{4}{-17}$$

$$\begin{aligned} &= -4 [3(-1) - 1(-2)] \\ &= 4 \end{aligned}$$

Question 2. (4 marks) Show that if A, B are invertible then $\det(A^{-1}BCAB^{-1}) = \det(C)$

$$\begin{aligned} \text{LHS} &= \det(A^{-1}BCAB^{-1}) \\ &= \det A^{-1} \det B \det C \det A \det B^{-1} \\ &= \frac{1}{\det A} \det B \det C \det A \frac{1}{\det B} \quad \text{since } A, B \text{ are invertible.} \\ &= \det C \\ &= \text{RHS.} \end{aligned}$$

Question 3. (5 marks)

a. (5 marks) Use the combinatorial definition of the determinant to compute:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

b. (2 marks) Justify the "visual way" of computing the determinant using part a.

a) Let $S = \{1, 2\}$

permutation of S	# of inversions	parity of permutation	elementary product	signed elementary product
(1, 2)	0	even	$a_{11} a_{22}$	$a_{11} a_{22}$
(2, 1)	1	odd	$a_{12} a_{21}$	$-a_{12} a_{21}$

$\therefore \det A = a_{11} a_{22} - a_{12} a_{21}$

b)

$$|A| = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

Question 4.

a. (5 marks) Find the inverse of the following matrix using the adjoint:

$$A = \begin{bmatrix} 0 & -2 \\ 3 & 1 \end{bmatrix}$$

$$a) A^{-1} = \frac{1}{\det A} \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^t = \frac{1}{6} \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix}^t$$

b. (2 marks) Using part a. solve the following equation:

$$Ax = b$$

where

$$b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\det A = 6$$

$$C_{11} = (-1)^{1+1} |1| = 1$$

$$C_{21} = (-1)^{2+1} |-2| = 2$$

$$C_{12} = (-1)^{1+2} |3| = -3$$

$$C_{22} = (-1)^{2+2} |0| = 0$$

$$\therefore A^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 2 \\ -3 & 0 \end{bmatrix}$$

b) $Ax = b$

$$A^{-1}Ax = A^{-1}b$$

$$x = A^{-1}b$$

$$\therefore x = \begin{bmatrix} 1/6 & 1/3 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/6 \\ -3/2 \end{bmatrix}$$

Question 5. (3 marks) For which value(s) of α is the following matrix invertible:

$$\begin{bmatrix} \alpha^2 - 1 & 0 & 0 \\ 1 & \alpha & 0 \\ 2 & -1 & \alpha + 1 \end{bmatrix}$$

a matrix is invertible if $\det \neq 0$

$$\begin{vmatrix} \alpha^2 - 1 & 0 & 0 \\ 1 & \alpha & 0 \\ 2 & -1 & \alpha + 1 \end{vmatrix} = (\alpha^2 - 1)(\alpha)(\alpha + 1) = (\alpha - 1)(\alpha)(\alpha + 1)^2$$

\therefore if $\alpha \neq 0, -1, 1$

Question 6. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 9 & 3 & 4 & 1 & 0 & 9 \\ 0 & 1 & 1 & 1 & 4 & 9 & 2 & 7 & 7 & 9 \\ 0 & 0 & 1 & 1 & 1 & 4 & 9 & 2 & 7 & 7 \\ 0 & 0 & 0 & 1 & 1 & 1 & 4 & 9 & 2 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 & 9 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, D = \begin{bmatrix} 2b - a & b \\ 2d - c & d \end{bmatrix}$$

a. (3 marks) If B is a 10×10 invertible matrix show that AB is invertible.

b. (4 marks) If $\det(D) = 2$ then find $\det(C)$.

a) Notice $\det A = 1 \neq 0$

$\therefore A$ is invertible

$$\det(AB) = \det A \det B \neq 0$$

since $\det B \neq 0$ because invertible

$\therefore AB$ is invertible.

b)

$$C \sim \begin{bmatrix} -a & b \\ -c & d \end{bmatrix} \sim \begin{bmatrix} 2b - a & b \\ 2d - c & d \end{bmatrix}$$

$-C_1$ $2C_2 + C_1 \rightarrow C_1$

$$(-1) \det C = \det D$$

$$(-1) \det C = 2$$

$$\det C = -2$$

Question 7. (5 marks) Compute the determinant of the following matrix using elementary operations

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -2 & 1 \\ 6 & 3 & 1 \end{bmatrix} \sim 3R_2 \begin{bmatrix} 3 & 1 & 1 \\ 6 & -6 & 3 \\ 6 & 3 & 1 \end{bmatrix} \sim \begin{matrix} -2R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{matrix} \begin{bmatrix} 3 & 1 & 1 \\ 0 & -8 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & -8 & 1 \\ 0 & 8 & -8 \end{bmatrix} \quad 8R_3$$

(3)(8) $\det A = \det B$

$$\det A = \frac{3(-8)(-7)}{3(8)}$$

$$\det A = \cancel{3} \cancel{8} \cancel{(-7)} \cancel{(-7)} \cancel{3} \cancel{8}$$

$$\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & -8 & 1 \\ 0 & 0 & -7 \end{bmatrix} \quad R_2 + R_3 \rightarrow R_3$$

$$\sim \begin{bmatrix} 3 & 1 & 1 \\ 0 & -8 & 1 \\ 0 & 0 & -7 \end{bmatrix} = B$$

Question 8. (5 marks) If A, B are 3×3 matrices, $\det(2A) = -8$ and $\det(B) = \sqrt{2}$ then find

$$\det((3AB)^t (2AB)^{-1} A^4 B^3 A^{-1}).$$

(show every step)

notice: $2^3 \det A = -8$
 $\det A = -1$

$$= \det(3AB)^t \det(2AB)^{-1} \det A^4 \det B^3 \det A^{-1}$$

$$= \det(3AB) \frac{1}{\det(2AB)} (\det A)^4 (\det B)^3 \frac{1}{\det A}$$

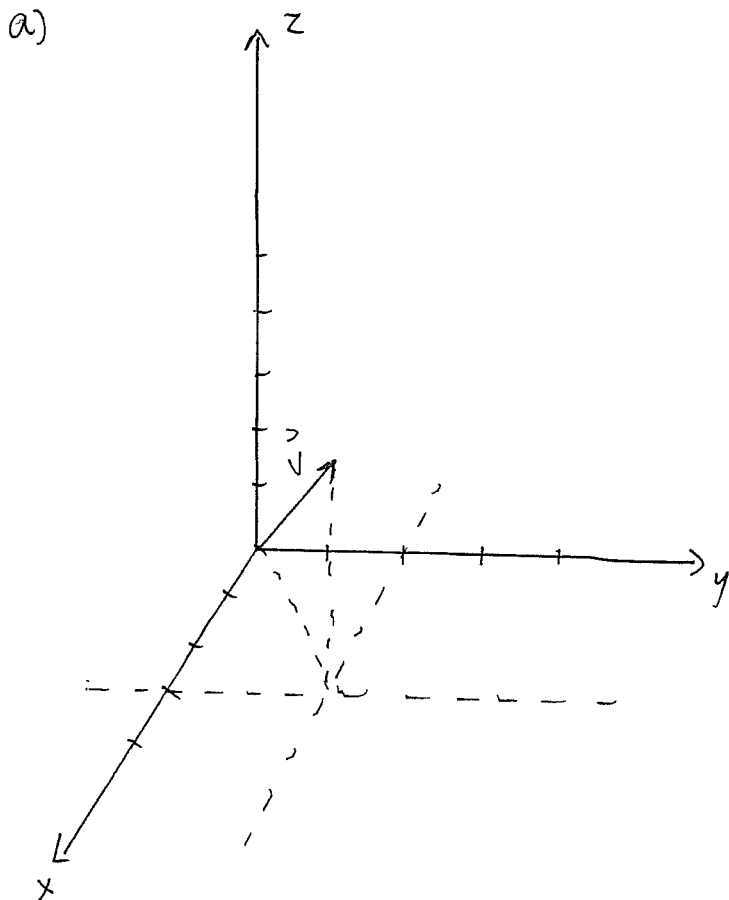
$$= 3^3 \cancel{\det A} \cancel{\det B} \frac{1}{2^3 \cancel{\det A} \cancel{\det B}} (\det A)^4 (\det B)^3 \frac{1}{\cancel{\det A}}$$

$$= \frac{3^3 (\sqrt{2})^3 (-1)^3}{2^3}$$

$$= -\frac{27\sqrt{2}}{4}$$

Question 9.

- (1 mark) Sketch the vector $\mathbf{v} = (3, 2, 4)$ on a right-handed coordinate system.
- (2 marks) Find a nonzero vector \mathbf{u} with terminal point $Q(3, 2, 1)$ which is oppositely directed to \mathbf{v} .
- (2 marks) Let $\mathbf{a} = (-1, 0, 2)$, $\mathbf{b} = (0, 3, 0)$ and $\mathbf{c} = (1, 1, 0)$. If $\mathbf{w} = 2(\mathbf{a} - \mathbf{b}) + 3\mathbf{c}$ then find a unit vector which has the same direction as \mathbf{w}



b) Let \vec{u} and \vec{v} have the same magnitude but different direction

$$\begin{aligned} \vec{u} &= -\vec{v} \\ \vec{PQ} &= -(3, 2, 4) \\ Q - P &= -(3, 2, 4) \\ P &= Q + (3, 2, 4) \\ P &= (3, 2, 1) + (3, 2, 4) \\ P &= (6, 4, 5) \end{aligned}$$

$$\begin{aligned} c) \vec{w} &= 2(\vec{a} - \vec{b}) + 3\vec{c} \\ &= 2(-1, 0, 2) - 2(0, 3, 0) + 3(1, 1, 0) \\ &= (1, -3, 4) \end{aligned}$$

\therefore a unit vector that has the same direction is

$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{(1, -3, 4)}{\sqrt{1^2 + (-3)^2 + 4^2}} = \frac{(1, -3, 4)}{\sqrt{26}}$$

Bonus Question. (3 marks) Consider

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix}$$

and use row reduction to show that $\det(A) = (b-a)(c-a)(c-b)$.

$$\sim \begin{array}{l} -aR_1 + R_2 \rightarrow R_2 \\ -a^2R_1 + R_3 \rightarrow R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & b^2-a^2 & c^2-a^2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & (b-a)(b+a) & (c-a)(c+a) \end{bmatrix}$$

$$\sim \begin{array}{l} -(b+a)R_2 + R_3 \rightarrow R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \\ 0 & 0 & (a-c)(b+a) + (c-a)(c+a) \end{bmatrix}$$

||
B

$$\det A = \det B$$

$$\det A = 1 \cdot (b-a) \left[(a-c)(b+a) + (c-a)(c+a) \right]$$

$$\det A = (b-a) \left[-(c-a)(b+a) + (c-a)(c+a) \right]$$

$$\det A = (b-a)(c-a) \left[-(b+a) + c+a \right]$$

$$\det A = (b-a)(c-a) \left[c-b \right]$$