Dawson College: **Linear Algebra**: 201-NYC-05-S07: Winter 2010

Name: Y. LAMONTAGNE
Student ID: ____________

Test 3

This test is graded out of 43 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Let $\mathbf{u} = (2,-1,0)$, $\mathbf{v} = (-4,0,2)$ and $\mathbf{w} = (2,1,3)$.

a. *(3 marks)* Find the angle between **u** and **v**.

b. *(3 marks)* Find a unit vector orthogonal to both **u** and **v**

c. *(3 marks)* Compute the scalar triple product of **u**, **v**, **w**.

d. *(1 mark)* Find the volume of the parallelepiped defined by **u**, **v**, **w**.

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$

$(2)(-4) + (-1)(0) + 0(2) = \sqrt{2^2+(-1)^2+0^2}\sqrt{(-4)^2+0^2+2^2}\cos\theta$

$-8 = \sqrt{5}\sqrt{20}\cos\theta$

$\frac{-8}{\sqrt{100}} = \cos\theta$

$\frac{-8}{10} = \cos\theta$

$\theta = \arccos\left(\frac{-4}{5}\right)$

$\theta \doteq 143°$

b) $\vec{u}\times\vec{v} = \left(\begin{vmatrix} -1 & 0 \\ 0 & 2 \end{vmatrix}, -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix}, \begin{vmatrix} 2 & -4 \\ -1 & 0 \end{vmatrix} \right)$

$\begin{matrix} 2 & -4 \\ -1 & 0 \\ 0 & 2 \end{matrix} = (-2,-4,-4)$

So $\frac{\vec{u}\times\vec{v}}{\|\vec{u}\times\vec{v}\|}$

$= \frac{(-2,-4,-4)}{\sqrt{(-2)^2+(-4)^2+(-4)^2}}$

$= \frac{(-2,-4,-4)}{\sqrt{36}}$

$= \left(\frac{-2}{6}, \frac{-4}{6}, \frac{-4}{6}\right)$

$= \left(\frac{-1}{3}, \frac{-2}{3}, \frac{-2}{3}\right)$

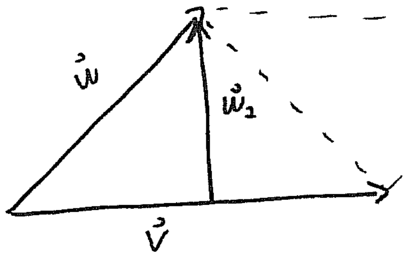
c) $\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 2 & -1 & 0 \\ -4 & 0 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 2\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} -4 & 2 \\ 2 & 3 \end{vmatrix}$

$= 2[-2] + [-12-4]$

$= -20$

d) The volume of the parallelepiped is $|-20| = 20$

Question 2. *(5 marks)* Using projections find the area of the triangle defined by $\mathbf{u} = (1,3,0)$ and $\mathbf{v} = (-2,0,-3)$.



where $\vec{w}_2 = \vec{u} - \text{proj}_{\vec{v}}\, \vec{u}$

$$\text{Area} = \frac{\|\vec{v}\|\|\vec{w}_2\|}{2} = \frac{\sqrt{(-2)^2+0^2+(-3)^2}\sqrt{\left(\frac{9}{13}\right)^2+\left(\frac{39}{13}\right)^2+\left(\frac{-6}{13}\right)^2}}{2}$$

$$= \frac{\sqrt{13}\sqrt{\frac{126}{13}}}{2} = \frac{\sqrt{126}}{2}$$

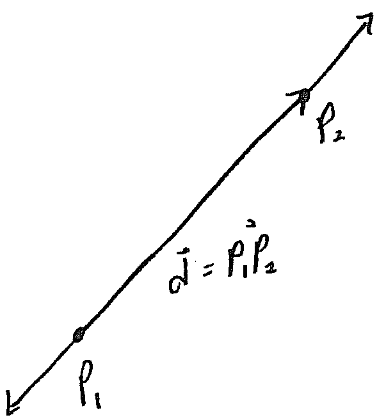
$$\vec{w}_2 = (1,3,0) - \frac{(1,3,0)\cdot(-2,0,-3)}{(-2,0,-3)\cdot(-2,0,-3)}(-2,0,-3)$$

$$= (1,3,0) - \frac{-2}{4+9}(-2,0,-3)$$

$$= (1,3,0) + \frac{2}{13}(-2,0,-3)$$

$$= \left(\frac{9}{13}, \frac{39}{13}, \frac{-6}{13}\right)$$

Question 3. *(2 marks)* Find the equation of the line passing through the points: $P_1(-2,1,3)$ and $P_2(1,-2,1)$.



$$\vec{d} = \vec{P_1P_2} = P_2 - P_1$$

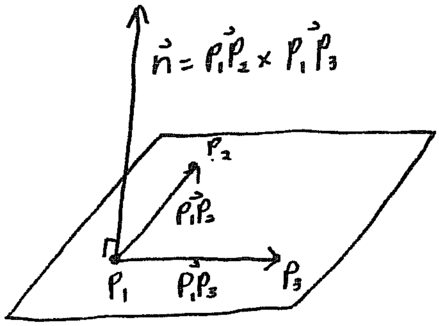
$$= (1,-2,1) - (-2,1,3)$$

$$= (3,-3,-2)$$

$$\therefore\ l: (x,y,z) = P_1 + t\vec{d}$$

$$= (-2,1,3) + t(3,-3,-2) \quad t \in \mathbb{R}$$

Question 4. *(3 marks)* Find the equation of the plane passing through the points: $P_1(7,-3,3)$, $P_2(2,-1,1)$ and $P_3(0,0,3)$.



$\vec{P_1P_2} = P_2 - P_1 = (2,-1,1) - (7,-3,3) = (-5,2,-2)$

$\vec{P_1P_3} = P_3 - P_1 = (0,0,3) - (7,-3,3) = (-7,3,0)$

$$\vec{n} = \vec{P_1P_2} \times \vec{P_1P_3} = \left(\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix}, - \begin{vmatrix} -5 & -7 \\ -2 & 0 \end{vmatrix}, \begin{vmatrix} -5 & -7 \\ 2 & 3 \end{vmatrix} \right)$$

$$\begin{matrix} -5 & -7 \\ 2 & 3 \\ -2 & 0 \end{matrix} \qquad = (6, 14, -1)$$

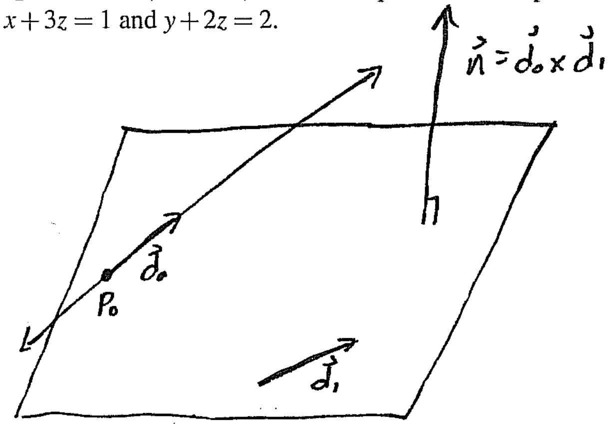
$\therefore\ 6x + 14y - z = d$

So $6(0) + 14(0) - 3 = d$

$-3 = d$

$\therefore\ 6x + 14y - z = -3$

Question 5. *(5 marks)* Find the equation of the plane which contains the line $(x,y,z) = \overbrace{(1,1,-1)}^{P_0} + t\overbrace{(1,3,4)}^{\vec{d}_0}$ parallel to and the intersection of $x+3z=1$ and $y+2z=2$.



Lets find the intersection of the two planes.

$$\begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix} \qquad \text{Let } z = t$$

$$\left. \begin{matrix} x + 3t = 1 \\ y + 2t = 2 \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} x = 1 - 3t \\ y = 2 - 2t \\ z = t \end{matrix} \right.$$

$\therefore\ l : (x,y,z) = \underbrace{(1,2,0)}_{P_1} + t\,\underbrace{(-3,-2,1)}_{\vec{d}_1}$

$$\vec{n} = \vec{d}_0 \times \vec{d}_1 = \left(\begin{vmatrix} 3 & -2 \\ 4 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & -3 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -3 \\ 3 & -2 \end{vmatrix} \right)$$

$$\begin{matrix} 1 & -3 \\ 3 & -2 \\ 4 & 1 \end{matrix} \qquad = (11, -13, 7)$$

$\therefore\ 11x - 13y + 7z = d$

So $11(1) - 13(2) + 7(0) = d$

$-15 = d$

$\therefore\ 11x - 13y + 7z = -15$

Question 6.

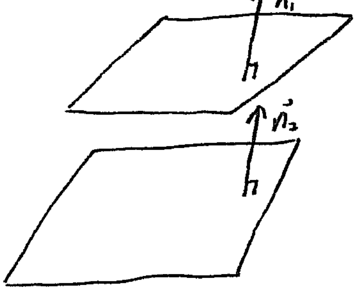
a. *(2 marks)* Determine if the two planes are parallel: $3x+y-z=10$ and $-9x-3y+3z=-101$.

b. *(2 marks)* Determine if the two planes are perpendicular: $2x-z=101$ and $y+z=-101$.

c. *(2 marks)* Determine if the line and the plane are perpendicular: $(x,y,z)=(1,2,2)+t(2,-1,2)$ and $\cancel{-2}6x+3y-6z=1$.

d. *(2 marks)* Does the point $(2,0,-3)$ lie on the line $(x,y,z)=(-1,-2,2)+t(6,4,-10)$.

a)



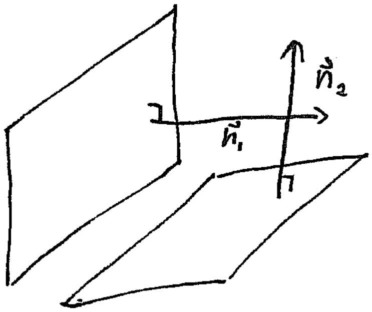
The two planes are parallel since

$\vec{n}_1 = (3,1,-1)$

$\vec{n}_2 = (-9,-3,3)$

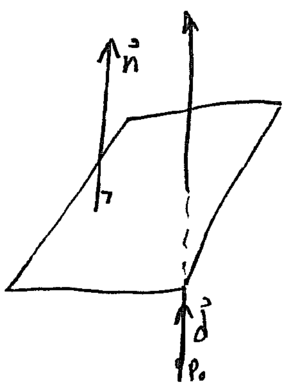
and $-3\vec{n}_1 = \vec{n}_2$

b)



The two planes are not perpendicular since $\vec{n}_1 \cdot \vec{n}_2 \neq 0$.

c)



The plane and line are perpendicular since $-3\vec{d} = \vec{n}$

d)

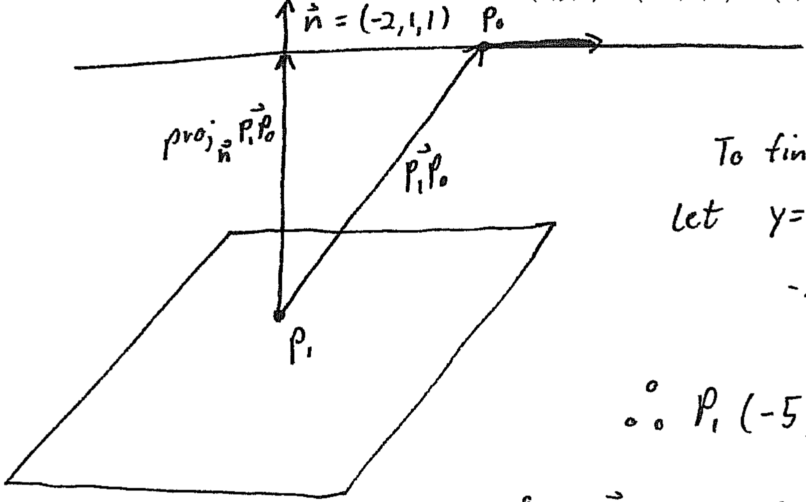
$(2,0,-3) = (-1,-2,2)+t(6,4,-10)$

$(3,2,-5) = t(6,4,-10)$

$t = \frac{1}{2}$

$\therefore$ $(2,0,-3)$ lies on the line.

Question 7. *(5 marks)* Find the distance between the line $(x,y,z) = \overbrace{(-2,2,3)}^{P_0} + t\overbrace{(3,0,6)}^{\vec{d}}$ and the plane $-2x+y+z=10$.



To find a point on the plane let $y=z=0$

$$-2x = 10$$
$$x = -5$$

$\therefore P_1(-5,0,0)$

$\therefore \vec{P_1P_0} = P_0 - P_1 = (-2,2,3) - (-5,0,0)$
$= (3,2,3)$

$$proj_{\vec{n}} \vec{P_1P_0} = \frac{(3,2,3)\cdot(-2,1,1)}{(-2,1,1)\cdot(-2,1,1)}(-2,1,1)$$

$$= \frac{3(-2)+2(1)+3(1)}{-2(-2)+1(1)+(1)(1)}(-2,1,1)$$

$$= \frac{-1}{6}(-2,1,1)$$

$$= \left(\frac{2}{6}, \frac{-1}{6}, \frac{-1}{6}\right)$$

$$\therefore d = \|proj_{\vec{n}} \vec{P_1P_0}\| = \sqrt{\left(\frac{2}{6}\right)^2 + \left(\frac{-1}{6}\right)^2 + \left(\frac{-1}{6}\right)^2}$$

$$= \sqrt{\frac{4+1+1}{36}}$$

$$= \sqrt{\frac{6}{36}}$$

$$= \sqrt{\frac{1}{6}}$$

Question 8. *(5 marks)* Maximize $P = 5x + 6y$ subject to

$$-2x + y \leq 40$$
$$3x - y \leq 10$$

$$\Rightarrow \begin{aligned} -2x + y + S_1 \quad\quad &= 40 \\ 3x - y \quad + S_2 \quad &= 10 \\ -5x - 6y \quad\quad + P &= 0 \end{aligned}$$

$$\begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 40 \\ 3 & -1 & 0 & 1 & 0 & 10 \\ -5 & -6 & 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{matrix} \\ R_1 + R_2 \to R_2 \\ 6R_1 + R_3 \to R_3 \end{matrix} \begin{bmatrix} -2 & 1 & 1 & 0 & 0 & 40 \\ 1 & 0 & 1 & 1 & 0 & 50 \\ -17 & 0 & 6 & 0 & 1 & 240 \end{bmatrix}$$

$$\begin{matrix} 2R_2 + R_1 \to R_1 \\ \\ 17R_2 + R_3 \to R_3 \end{matrix} \begin{bmatrix} 0 & 1 & 3 & 2 & 0 & 140 \\ 1 & 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & 23 & 17 & 1 & 1090 \end{bmatrix}$$

$\therefore$ max is 1090 at $x = 50$, $y = 140$.

Bonus Question. *(3 marks)* Prove: If **u** and **a** are non-zero vectors then

$$\|\text{proj}_{\mathbf{a}}\,\mathbf{u}\| = \|\mathbf{u}\|\,|\cos\theta|$$

where θ is the angle between **u** and **a**.

$$\|\text{proj}_{\vec{a}}\,\vec{u}\| = \left\| \frac{\vec{u}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\,\vec{a} \right\| = \left| \frac{\vec{u}\cdot\vec{a}}{\vec{a}\cdot\vec{a}} \right| \|\vec{a}\|$$

$$= \frac{\left| \|\vec{u}\|\cancel{\|\vec{a}\|}\cos\theta \right|}{\cancel{\|\vec{a}\|^2}} \cancel{\|\vec{a}\|}$$

$$= \|\vec{u}\|\,|\cos\theta|$$