Dawson College: **Linear Algebra**: 201-NYC-05-S07: Winter 2010

Name: Y. LAMONTAGNE
Student ID: ____________

Test 3

This test is graded out of 50 marks. No books, notes, graphing calculators or cell phones are allowed. You must show all your work, the correct answer is worth 1 mark the remaining marks are given for the work. If you need more space for your answer use the back of the page.

Question 1. Let $\mathbf{u} = (2,0,-1)$, $\mathbf{v} = (-4,2,0)$ and $\mathbf{w} = (1,2,3)$.

a. *(3 marks)* Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

b. *(3 marks)* Find a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$

c. *(3 marks)* Compute the scalar triple product of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

d. *(1 mark)* Find the volume of the parallelepiped defined by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

a) $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta$

$2(-4)+0(2)+(-1)(0) = \sqrt{2^2+0+(-1)^2}\sqrt{(-4)^2+2^2+0^2}\cos\theta$

$-8 = \sqrt{5}\sqrt{20}\cos\theta$

$-8 = \sqrt{100}\cos\theta$

$\frac{-8}{10} = \cos\theta$

$\frac{-4}{5} = \cos\theta$

$\theta = \arccos\left(\frac{-4}{5}\right)$

$\theta \doteq 143^\circ$

b) $\vec{u}\times\vec{v} = \left(\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}, -\begin{vmatrix} 2 & -4 \\ -1 & 0 \end{vmatrix}, \begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} \right)$

$\begin{matrix} 2 & -4 \\ 0 & 2 \\ -1 & 0 \end{matrix}$

$= (2, 4, 4)$

So $\frac{(2,4,4)}{\|(2,4,4)\|}$

$= \frac{(2,4,4)}{\sqrt{2^2+4^2+4^2}}$

$= \frac{(2,4,4)}{6} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

c) $\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} 2 & 0 & -1 \\ -4 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} -4 & 2 \\ 1 & 2 \end{vmatrix}$

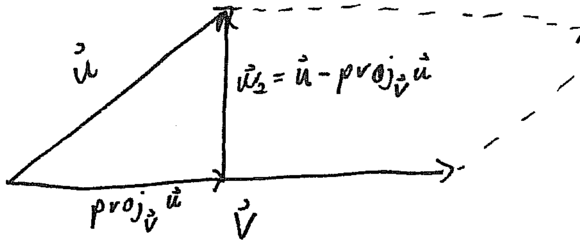
$= 2[6] - [-4(2) - 2(1)]$

$= 12 + 10$

$= 22$

d) $\therefore$ the volume of the parallelepiped 22.

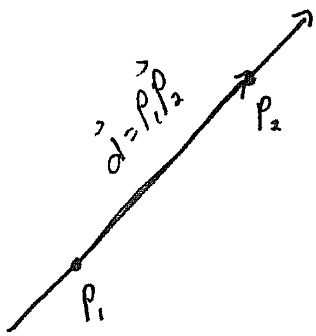
Question 2. *(5 marks)* Using projections find the area of the parrallelogram defined by $\mathbf{u} = (1,0,3)$ and $\mathbf{v} = (0,-2,-3)$.



$$\text{Area} = \text{base} \times \text{height}$$
$$= \|\vec{v}\| \, \|\vec{w}_2\| = \sqrt{0^2+(-2)^2+(-3)^2}\sqrt{\left(\frac{13}{13}\right)^2+\left(\frac{-18}{13}\right)^2+\left(\frac{12}{13}\right)^2}$$
$$= \sqrt{13}\sqrt{\frac{637}{13^2}}$$
$$= 7$$

$$\vec{w}_2 = (1,0,3) - \frac{(1,0,3)\cdot(0,-2,-3)}{(0,-2,-3)\cdot(0,-2,-3)}(0,-2,-3)$$
$$= (1,0,3) - \frac{-9}{13}(0,-2,-3)$$
$$= \left(1, -\frac{18}{13}, \frac{12}{13}\right)$$

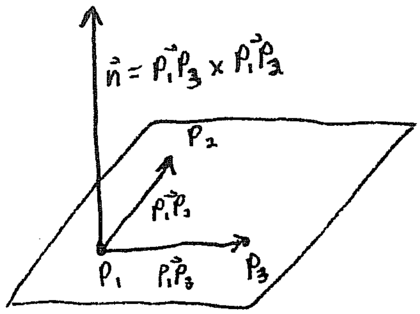
Question 3. *(2 marks)* Find the equation of the line passing through the points: $P_1(2,-3,4)$ and $P_2(2,-1,1)$.



$$\vec{d} = \vec{P_1P_2} = P_2 - P_1 = (2,-1,1) - (2,-3,4)$$
$$= (0,2,-3)$$

$$\therefore \ \ell : (x,y,z) = P_1 + t\vec{d}$$
$$= (2,-3,4) + t\,(0,2,-3) \qquad t \in \mathbb{R}$$

Question 4. *(3 marks)* Find the equation of the plane passing through the points: $P_1(8,-3,4)$, $P_2(2,1,2)$ and $P_3(3,0,0)$.



$$\vec{P_1P_3} = P_3 - P_1 = (3,0,0) - (8,-3,4) = (-5,3,-4)$$
$$\vec{P_1P_2} = P_2 - P_1 = (2,1,2) - (8,-3,4) = (-6,4,-2)$$
$$\vec{n} = \vec{P_1P_3} \times \vec{P_1P_2} = \left(\begin{vmatrix} 3 & 4 \\ -4 & -2 \end{vmatrix}, -\begin{vmatrix} -5 & -6 \\ -4 & -2 \end{vmatrix}, \begin{vmatrix} -5 & -6 \\ 3 & 4 \end{vmatrix} \right)$$

$$\begin{matrix} -5 & -6 \\ 3 & 4 \\ -4 & -2 \end{matrix}$$

$$= (10, 14, -2)$$

$\therefore$ $10x + 14y - 2z = d$

$10(3) + 14(0) - 2(0) = d$

$30 = d$

$\therefore$ $10x + 14y - 2z = 30$

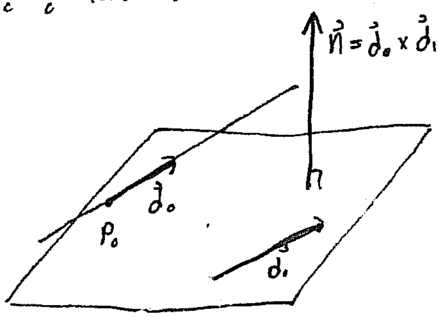
Question 5. *(5 marks)* Find the equation of the plane which contains the line $(x,y,z) = \underbrace{(1,0,-2)}_{P_0} + t\underbrace{(2,3,4)}_{\vec{d_0}}$ and the intersection of $x+5z = 1$ and $y+4z = 3$.

parallel to

Lets find the intersection

$$\left[\begin{matrix} 1 & 0 & 5 & 1 \\ 0 & 1 & 4 & 3 \end{matrix}\right]$$ let $z = t$ So $\left.\begin{matrix} x + 5t = 1 \\ y + 4t = 3 \\ z = t \end{matrix}\right\} \Leftrightarrow \left\{\begin{matrix} x = 1 - 5t \\ y = 3 - 4t \\ z = t \end{matrix}\right.$

$\therefore$ intersection at $\ell : (x,y,z) = \underbrace{(1,3,0)}_{P_1} + t\underbrace{(-5,-4,1)}_{\vec{d_1}}$ $t \in \mathbb{R}$



$$\vec{n} = \vec{d_0} \times \vec{d_1} = \left(\begin{vmatrix} 3 & -4 \\ 4 & 1 \end{vmatrix}, -\begin{vmatrix} 2 & -5 \\ 4 & 1 \end{vmatrix}, \begin{vmatrix} 2 & -5 \\ 3 & -4 \end{vmatrix} \right)$$

$$\begin{matrix} 2 & -5 \\ 3 & -4 \\ 4 & 1 \end{matrix}$$

$$= (19, -22, 7)$$

$\therefore$ $19x - 22y + 7z = d$

So $19(1) - 22(0) + 7(-2) = d$

$19 - 14 = d$

$5 = d$

$\therefore$ $19x - 22y + 7z = 5$

Question 6.

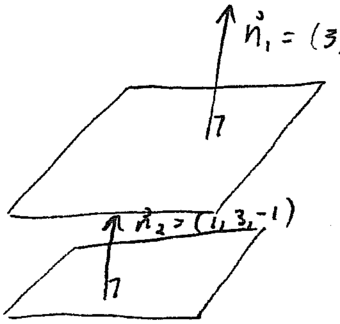
a. *(2 marks)* Determine if the two planes are parallel: $3x+y-z=10$ and $x+3y-z=-101$.

b. *(2 marks)* Determine if the two planes are perpendicular: $2x-z=101$ and $y=-101$.

c. *(2 marks)* Determine if the line and the plane are perpendicular: $(x,y,z)=(1,2,2)+t(2,-1,2)$ and $-6x+3y-6z=1$.

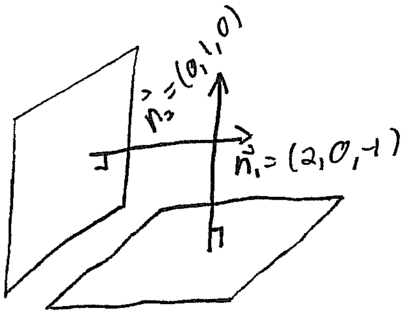
d. *(2 marks)* Does the point $(2,0,-4)$ lie on the line $(x,y,z)=(-1,-2,2)+t(6,+4,-12)$.

a)



The two planes are not parallel since the normals are not a multiple of each other.

b)

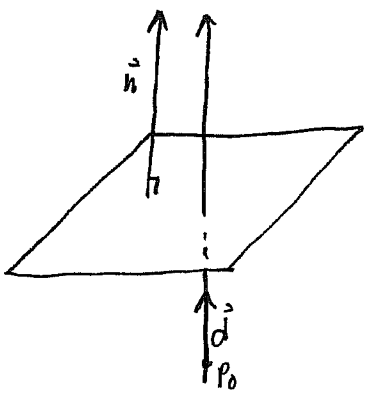


The two planes are perpendicular if $\vec{n}_2 \cdot \vec{n}_1 = 0$.

$\vec{n}_1 \cdot \vec{n}_2 = (0,1,0)\cdot(2,0,-1) = 0$

$\therefore$ the two planes are perpendicular

c)



The plane and line are perpendicular if the normal is a multiple of the direction vector

$-3\vec{d} = \vec{n}$

$\therefore$ the plane and line are perpendicular.

d)

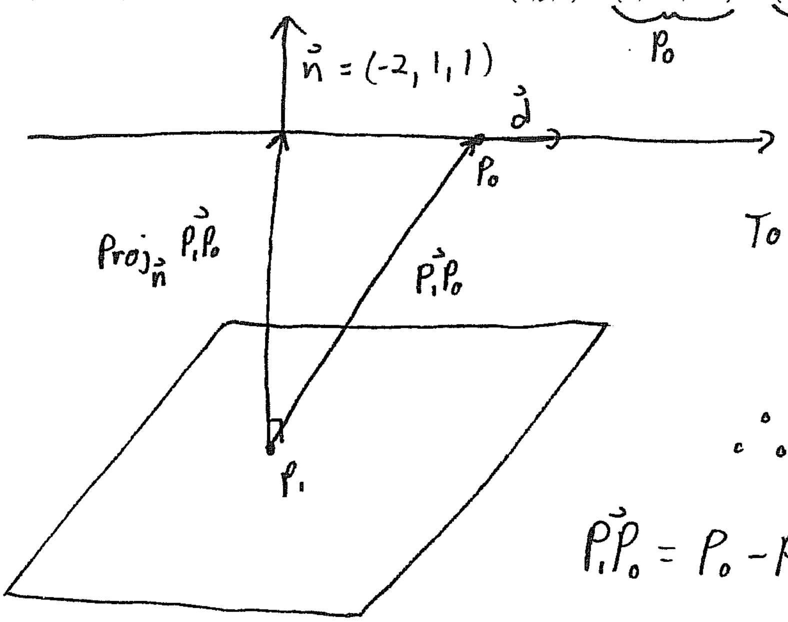
$(2,0,-4) = (-1,-2,2) + t(6,+4,-12)$

$t(6,+4,-12) = (3,2,-6)$

$\therefore t = \frac{1}{2}$

$\therefore$ The point is on the line.

Question 7. *(5 marks)* Find the distance between the line $(x,y,z) = \underbrace{(1,-2,-4)}_{P_0} + t\underbrace{(2,0,4)}_{\vec{d}}$ and the plane $-2x+y+z=6$.



To find P_1, let $y=z=0$

$-2x+0+0=6$

$x=-3$

$\therefore P_1(-3,0,0)$

$\vec{P_1P_0} = P_0 - P_1 = (1,-2,-4)-(-3,0,0)$
$= (4,-2,-4)$

$$\text{proj}_{\vec{n}} \vec{P_1P_0} = \frac{(4,-2,-4)\cdot(-2,1,1)}{(-2,1,1)\cdot(-2,1,1)}(-2,1,1)$$

$$= \frac{4(-2)+(-2)(1)+(-4)(1)}{(-2)(-2)+(1)(1)+(1)(1)}(-2,1,1)$$

$$= \frac{-8-2-4}{4+1+1}(-2,1,1)$$

$$= \frac{-14}{6}(-2,1,1)$$

$$= \frac{-7}{3}(-2,1,1)$$

$$= \left(\frac{14}{3}, \frac{-7}{3}, \frac{-7}{3}\right)$$

$$d = \|\text{proj}_{\vec{n}} \vec{P_1P_0}\| = \left\|\left(\frac{14}{3}, \frac{-7}{3}, \frac{-7}{3}\right)\right\| = \sqrt{\left(\frac{14}{3}\right)^2 + \left(\frac{-7}{3}\right)^2 + \left(\frac{-7}{3}\right)^2}$$

$$= \sqrt{\frac{196+49+49}{9}}$$

$$= \sqrt{\frac{98}{3}}$$

Question 8. *(5 marks)* Maximize $P = 4x + 5y$ subject to

$$-x + y \leq 40$$
$$2x - y \leq 10$$

$$\begin{aligned} -x + y + s_1 \quad\quad &= 40 \\ 2x - y \quad + s_2 \quad &= 10 \\ -4x - 5y \quad\quad + P &= 0 \end{aligned}$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 40 \\ 2 & -1 & 0 & 1 & 0 & 10 \\ -4 & -5 & 0 & 0 & +1 & 0 \end{bmatrix}$$

$$\begin{array}{l} \\ R_1 + R_2 \to R_2 \\ 5R_1 + R_3 \to R_3 \end{array} \begin{bmatrix} -1 & 1 & 1 & 0 & 0 & 40 \\ 1 & 0 & 1 & 1 & 0 & 50 \\ -9 & 0 & 5 & 0 & 1 & 200 \end{bmatrix}$$

$$\begin{array}{l} R_2 + R_1 \to R_1 \\ \\ 9R_2 + R_3 \to R_3 \end{array} \begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 90 \\ 1 & 0 & 1 & 1 & 0 & 50 \\ 0 & 0 & 14 & 9 & 1 & 650 \end{bmatrix}$$

$\therefore$ max is 650 at $x = 50$, $y = 90$.

Bonus Question. *(3 marks)* Prove: If **u** and **a** are non-zero vectors then

$$\|\text{proj}_{\mathbf{a}}\, \mathbf{u}\| = \|\mathbf{u}\|\, |\cos\theta|$$

where θ is the angle between **u** and **a**.

$$\|\text{proj}_{\vec{a}}\, \vec{u}\| = \left\| \frac{\vec{u} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\, \vec{a} \right\| = \left| \frac{\vec{u} \cdot \vec{a}}{\vec{a} \cdot \vec{a}} \right| \|\vec{a}\|$$

$$= \frac{|\vec{u} \cdot \vec{a}|}{\|\vec{a}\|^2} \|\vec{a}\|$$

$$= \frac{|\, \|\vec{u}\| \, \|\vec{a}\| \cos\theta \,|}{\|\vec{a}\|^2} \|\vec{a}\|$$

$$= \|\vec{u}\| |\cos\theta|$$