

Solve

$$2x_1 + x_2 + x_4 + x_5 = 1$$

$$3x_1 + 3x_3 + x_4 = 2$$

$$5x_1 + 3x_2 + 4x_4 - x_5 = 3$$

Using Gaussian elimination and back substitution.

$$\begin{bmatrix} 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 3 & 1 & 0 & 2 \\ 5 & 3 & 0 & 4 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{matrix} 2R_2 \\ 2R_3 \end{matrix} \begin{bmatrix} 2 & 1 & 0 & 1 & 1 & 1 \\ 6 & 0 & 6 & 2 & 0 & 4 \\ 10 & 6 & 0 & 8 & -2 & 6 \end{bmatrix}$$

$$\sim \begin{matrix} -3R_1 + R_2 \rightarrow R_2 \\ -5R_1 + R_3 \rightarrow R_3 \end{matrix} \begin{bmatrix} 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & -3 & 6 & -1 & -3 & 1 \\ 0 & 1 & & -2 & -7 & 1 \end{bmatrix}$$

$$\sim \begin{matrix} -3R_3 \end{matrix} \begin{bmatrix} 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & -3 & 6 & -1 & -3 & 1 \\ 0 & -3 & 0 & 6 & 21 & -3 \end{bmatrix}$$

$$\sim \begin{matrix} -R_2 + R_3 \rightarrow R_3 \end{matrix} \begin{bmatrix} 2 & 1 & 0 & 1 & 1 & 1 \\ 0 & -3 & 6 & -1 & -3 & 1 \\ 0 & 0 & -6 & 7 & 20 & -4 \end{bmatrix}$$

$$\sim \begin{matrix} \frac{1}{2}R_1 \\ \frac{1}{3}R_2 \\ \frac{1}{6}R_3 \end{matrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -2 & \frac{1}{3} & 1 & -\frac{1}{3} \\ 0 & 0 & 1 & -\frac{7}{6} & -\frac{20}{6} & \frac{4}{6} \end{bmatrix}$$

x_4 and x_5 are free variables.

$$\therefore \begin{aligned} x_4 &= t \\ x_5 &= s \end{aligned}$$

so we sub in the system.

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_4 + \frac{1}{2}x_5 &= \frac{1}{2} \\ x_2 - 2x_3 + \frac{1}{3}x_4 + x_5 &= -\frac{1}{3} \\ x_3 - \frac{7}{6}x_4 - \frac{20}{6}x_5 &= \frac{4}{6} \end{aligned}$$

and get

$$\begin{aligned} x_1 + \frac{1}{2}x_2 + \frac{1}{2}t + \frac{1}{2}s &= \frac{1}{2} \quad (1) \\ x_2 - 2x_3 + \frac{1}{3}t + s &= -\frac{1}{3} \quad (2) \\ x_3 - \frac{7}{6}t - \frac{20}{6}s &= \frac{4}{6} \quad (3) \end{aligned}$$

Solve for x_3 eqn (3)

$$x_3 = \frac{4}{6} + \frac{7}{6}t + \frac{20}{6}s \quad (4)$$

sub (4) into (2)

$$x_2 - 2\left(\frac{4}{6} + \frac{7}{6}t + \frac{20}{6}s\right) + \frac{1}{3}t + s = -\frac{1}{3}$$

$$x_2 = -\frac{1}{3} + 4 + \frac{7}{3}t + \frac{20}{3}s - \frac{1}{3}t - s$$

$$x_2 = \frac{11}{3} + 2t - \frac{17}{3}s \quad (5)$$

sub (5) into (1)

$$x_1 + \frac{1}{2}\left(\frac{11}{3} + 2t - \frac{17}{3}s\right) + \frac{1}{2}t + \frac{1}{2}s = \frac{1}{2}$$

$$x_1 + \frac{11}{6} + t - \frac{17}{6}s + \frac{1}{2}t + \frac{1}{2}s = \frac{1}{2}$$

$$X_1 = \frac{1}{2} - \frac{11}{6} - t + \frac{17}{6}s - \frac{1}{2}t - \frac{1}{2}s$$

$$X_1 = \frac{-4}{3} - \frac{3}{2}t + \frac{7}{3}s$$

∴ the general solution is

$$X_1 = \frac{-4}{3} - \frac{3}{2}t + \frac{7}{3}s$$

$$X_2 = \frac{11}{3} + 2t - \frac{17}{3}s$$

$$X_3 = \frac{4}{6} + \frac{7}{6}t + \frac{20}{6}s$$

$$X_4 = t$$

$$X_5 = s$$