NAME: SOLUTIONS

TEST 1

DAWSON COLLEGE

NYA-Electrotech Section 7 - Calculus 1

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This test is marked out of **65 points**

Question 1. *(4 marks each)*
Find the derivative of each of the following functions. For each function, indicate which of the rules (if any) you have used among: the product rule, quotient rule, chain rule as well as how many times you have used each.

(a) $f(x) = 3(\ln x)(-3x^4 + 2x^2 - \frac{1}{x^2})$ NOTE $\frac{1}{x^2} = x^{-2}$ derivative $-2x^{-3}$

PRODUCT RULE

$$f'(x) = \frac{3}{x}\left(-3x^4 + 2x^2 - \frac{1}{x^2}\right) + \left(-12x^3 + 4x + 2x^{-3}\right)(3\ln x)$$

$$= \boxed{-9x^3 + 6x - \frac{3}{x^3} + \left(-36x^3 + 12x + \frac{6}{x^3}\right)\ln x}$$

(b) $g(t) = \cos\left(\frac{2t-4t^2}{e^t}\right)$

CHAIN rule
QUOTIENT RULE

$$g'(t) = -\sin\left(\frac{2t-4t^2}{e^t}\right) \cdot \left(\frac{(2-8t)e^t - e^t(2t-4t^2)}{(e^t)^2}\right)$$

$$= \left[-\sin\left(\frac{2t-4t^2}{e^t}\right)\right]\left[\frac{e^t(2-8t-2t+4t^2)}{(e^t)^2}\right]$$

$$= \boxed{\left[-\sin\left(\frac{2t-4t^2}{e^t}\right)\right]\left[\frac{2-10t+4t^2}{e^t}\right]}$$

(c) $f(t) = -\frac{1}{5t^5} + \pi^3 - \frac{7t^4}{t^{-6}} + 2\sqrt{t}$

$= -\frac{1}{5}t^{-5} + \pi^3 - 7t^{10} + 2\,t^{1/2}$

$= \boxed{t^{-6} - 70t^9 + t^{-1/2}}$

(d) $h(x) = (\sin(3x))(\cos(-4x))$

$\boxed{\text{product RuLe} \quad \text{2 CHAIN RULES}}$

$h'(x) = \cos(3x)\cdot 3\cos(-4x)$
$\quad + (-\sin(-4x))\cdot(-4)\cdot(\sin 3x)$

$= \boxed{3\cos(3x)\cos(-4x) + 4\sin(-4x)\sin(3x)}$

(e) $g(t) = \sqrt{\ln(2t^5 - t^2)} = \left(\ln(2t^5 - t^2)\right)^{1/2}$

$\boxed{\text{CHAIN rule inside A CHAIN rule}}$

$\boxed{g'(t) = \frac{1}{2}\left[\ln(2t^5 - t^2)\right]^{-1/2} \cdot \left[\frac{1}{2t^5 - t^2}\right] \cdot \left[(10t^4 - 2t)\right]}$

(f) $g(x) = \frac{1}{2}e^{(\tan x)(x^2 - 2x)}$

$\boxed{\text{product inside CHAIN rule}}$

$\boxed{g'(x) = \frac{1}{2}e^{(\tan x)(x^2 - 2x)} \cdot \left[\sec^2 x\,(x^2 - 2x) + (2x - 2)(\tan x)\right]}$

Question 2. *(6 marks)*
Consider the tangent lines to the curves $y_1 = \sin(\pi e^x)$ and $y_2 = \frac{1}{8\pi}(2x+2)^2$ at $x = 0$. Determine if these tangent lines are parallel, perpendicular or neither?

WE CAN FIND THE SLOPES OF THE TWO LINES AT $x = 0$

$m_1 = y_1' = \cos(\pi e^x) \cdot \pi e^x$

AT $x=0] = \cos(\pi) \cdot \pi$
$= -\pi$

$m_2 = y_2' = \frac{2}{8\pi}(2x+2) \cdot 2$

$= \frac{2x+2}{2\pi}$

AT $x=0] = \frac{2}{2\pi} = \frac{1}{\pi}$

SINCE $m_1 \cdot m_2 = -\pi \cdot \frac{1}{\pi} = -1$

THE TANGENT LINES ARE $\boxed{\text{perpendicular}}$.

Question 3. *(4 marks)*
Find the derivative of $f(x) = 2x \cos x \ln x$

TWO PRODUCT RULES (ONE INSIDE THE OTHER)

$$\boxed{f'(x) = [2\cos x - 2x\sin x]\ln x + \frac{1}{x}(2x\cos x)}$$

Question 4. *(3 marks each)*
Evaluate the following limits algebraically (not using a table of values). If the limit does not exist; write DNE and determine what the value of the one-sided limits tend to (in this case you can evaluate the one-sided limits numerically using a table of values).

(a) $\lim_{x\to 4} \frac{2-\sqrt{x}}{x^2-16}$

$$= \lim_{x\to 4} \frac{2-\sqrt{x}}{x^2-16} \frac{(2+\sqrt{x})}{(2+\sqrt{x})}$$

$$= \lim_{x\to 4} \frac{4-x}{(x-4)(x+4)(2+\sqrt{x})}$$

$$= \lim_{x\to 4} \frac{-\cancel{(x-4)}}{\cancel{(x-4)}(x+4)(2+\sqrt{x})}$$

$$= \frac{-1}{8\cdot(4)} = \boxed{\frac{-1}{32}}$$

(b) $\lim_{x\to\infty} \frac{4x^4-5x^2+x-5}{2x^4+3x^2-7x}$

$$= \lim_{x\to\infty} \frac{\frac{4x^4}{x^4} - \frac{5x^2}{x^4} + \frac{x}{x^4} - \frac{5}{x^4}}{\frac{2x^4}{x^4} + \frac{3x^2}{x^4} - \frac{7x}{x^4}}$$

$$= \lim_{x\to\infty} \frac{4 - \cancel{\frac{5}{x^2}}^{\,0} + \cancel{\frac{1}{x^3}}^{\,0} - \cancel{\frac{5}{x^4}}^{\,0}}{2 + \cancel{\frac{3}{x^2}}^{\,0} - \cancel{\frac{7}{x^3}}^{\,0}} = 4/2 = \boxed{2}$$

(c) $\lim_{x\to -3} \frac{2x+6}{9-x^2}$

$$= \lim_{x\to -3} \frac{2\cancel{(x+3)}}{(3-x)\cancel{(3+x)}}$$

$$= \frac{2}{6} = \boxed{\frac{1}{3}}$$

Question 6. *(5 marks)*
Use the limit definition of the derivative to find the derivative of the function $f(x) = -2x + \frac{1}{x}$. Note that no marks will be given if you do not use the limit definition to evaluate the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(-2(x+h) + \frac{1}{x+h}\right) - \left(-2x + \frac{1}{x}\right)}{h}$$

$$= \lim_{h \to 0} \frac{-2x - 2h + \frac{1}{x+h} + 2x - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \left[-2h + \frac{1}{x+h} - \frac{1}{x}\right] \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \left[-2h + \left(\frac{x - (x+h)}{(x+h)x}\right)\right] \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \left[-2h \frac{-h}{(x+h)x}\right] \frac{1}{h} = \lim_{h \to 0} -2 - \frac{1}{(x+h)x}$$

$$= \boxed{-2 - \frac{1}{x^2}}$$

Question 7. *(4 marks)*
Find the value of the constant a if the slope of the tangent line to the curve $y = -3ax^2 + 3x + 2$ at $x = -1$ is equal to 6.

slope $= y' = -6ax + 3$

AT $x = -1 \implies m = 6a + 3$

AT $x = -1$ $m = 6$

so $6a + 3 = 6$

$6a = 3$

$\boxed{a = \frac{1}{2}}$

(d) $\lim_{x\to3}\frac{2x+6}{9-x^2} = \lim_{x\to3}\frac{2\cancel{(x+3)}}{(3-x)\cancel{(3+x)}}$

$= \lim_{x\to3}\frac{2}{3-x}$ WE CANNOT proceed

$\boxed{\text{DNE}}$ WE EVALUATE ONE-SIDED LIMITS

FROM LEFT

x	2.9	2.99	2.999
$\frac{2}{3-x}$	20	200	2000

$\boxed{\lim_{x\to3^-}\frac{2x+6}{9-x^2} \text{ tends to } +\infty}$

FROM RIGHT

x	3.1	3.01	3.001
$\frac{2}{3-x}$	-20	-200	-2000

$\boxed{\lim_{x\to3^+}\frac{2x+6}{9-x^2} \text{ tends to } -\infty}$

Question 5. *(5 marks)*
Determine if the function $g(x)$ is continuous at $x=-2$. **Justify** your answer using the 3 conditions necessary for continuity at a point.

$$g(x) = \begin{cases} 3x^2-4x & x>-2 \\ -12x-4 & x<-2 \\ 20 & x=-2 \end{cases}$$

1. $g(-2)$ EXISTS ; $\boxed{g(-2) = 20}$

2. $\lim_{x\to-2^-} g(x) = \lim_{x\to-2^-} -12x-4 = 24-4 = 20$

$\lim_{x\to-2^+} g(x) = \lim_{x\to-2^+} 3x^2-4x = 3(-2)^2-4(-2)$
$= 12+8 = 20$

$\boxed{\lim_{x\to-2} g(x) = 20}$

3. SINCE $\lim_{x\to-2} g(x) = g(-2) = 20$

THE FUNCTION IS CONTINUOUS.

Question 8. *(5 marks)*
The distance s (in metres) traveled by a metro train after the breaks are applied is given by $s = 20t - 2t^2$, where t is the time (in seconds).
(a) How far has the train traveled 2 seconds after the breaks are applied?
(b) How much time elapses between the moment the brakes are applied and the moment the train stops?
(c) How far has the train traveled between applying the brakes and coming to a stop?

(a) $S = 20(2) - 2(2)^2 = 40 - 8 = 32\text{M}$

(b) WHEN TRAIN STOPS Velocity $V = 0$

$V = S' = 20 - 4t \Rightarrow 20 - 4t = 0$

$20 = 4t$

$t = 5$

$\boxed{\text{5 seconds elapse}}$

(c) $S(5) = 20(5) - 2(5)^2 = 100 - 50 = 50\text{M}$

THE TRAIN TRAVELS $\boxed{50\text{M}}$ BEFORE COMING TO A STOP

BONUS. *(3 marks)*
Explain in your own words how this series of diagrams illustrates the concept of the derivative.