NAME: SOLUTIONS

TEST 3

DAWSON COLLEGE

NYA-Electrotech Section 7 - Calculus 1

Instructor: E. Richer

Date: May 7th 2010

This test is marked out of **75 points**

Question 1. *(10 marks)*

Integrate each of the given expressions. *(1 mark each)*

(a) $\int \sin x \, dx = -\cos x + C$

(b) $\int \frac{2}{\sqrt{1-4x^2}} \, dx = \arcsin(2x) + C$

(c) $\int \frac{1}{x} \, dx = \ln|x| + C$

(d) $\int \frac{1}{(\ln 5)x} \, dx = \log_5|x| + C$

(e) $\int (\ln 2)2^x \, dx = 2^x + C$

(f) $\int \sec^2 x \, dx = \tan x + C$

(g) $\int 3\cos(3x) \, dx = \sin(3x) + C$

(h) $\int -e^{-x} \, dx = e^{-x} + C$

(i) $\int \frac{5}{1+25x^2} \, dx = \arctan(5x) + C$

(j) $\int \frac{1}{2\sqrt{x}} \, dx = \sqrt{x} + C$

Question 2 *(5 marks)*

Find the function $y = f(x)$ that has the following properties:

- Its second derivative is $2 + e^x - \sin x$
- The slope of the tangent line to the curve at $x = 0$ is -4
- It passes through the point $(0, 1)$.

① $f'(x) = \int 2 + e^x - \sin x \, dx$

$= 2x + e^x + \cos x + C$

$-4 = 2(0) + e^0 + \cos(0) + C$

$-4 = 2 + C$

$C = -6$ so $f'(x) = 2x + e^x + \cos x - 6$

② $f(x) = \int 2x + e^x + \cos x - 6 \, dx$

$= x^2 + e^x + \sin x - 6x + C$

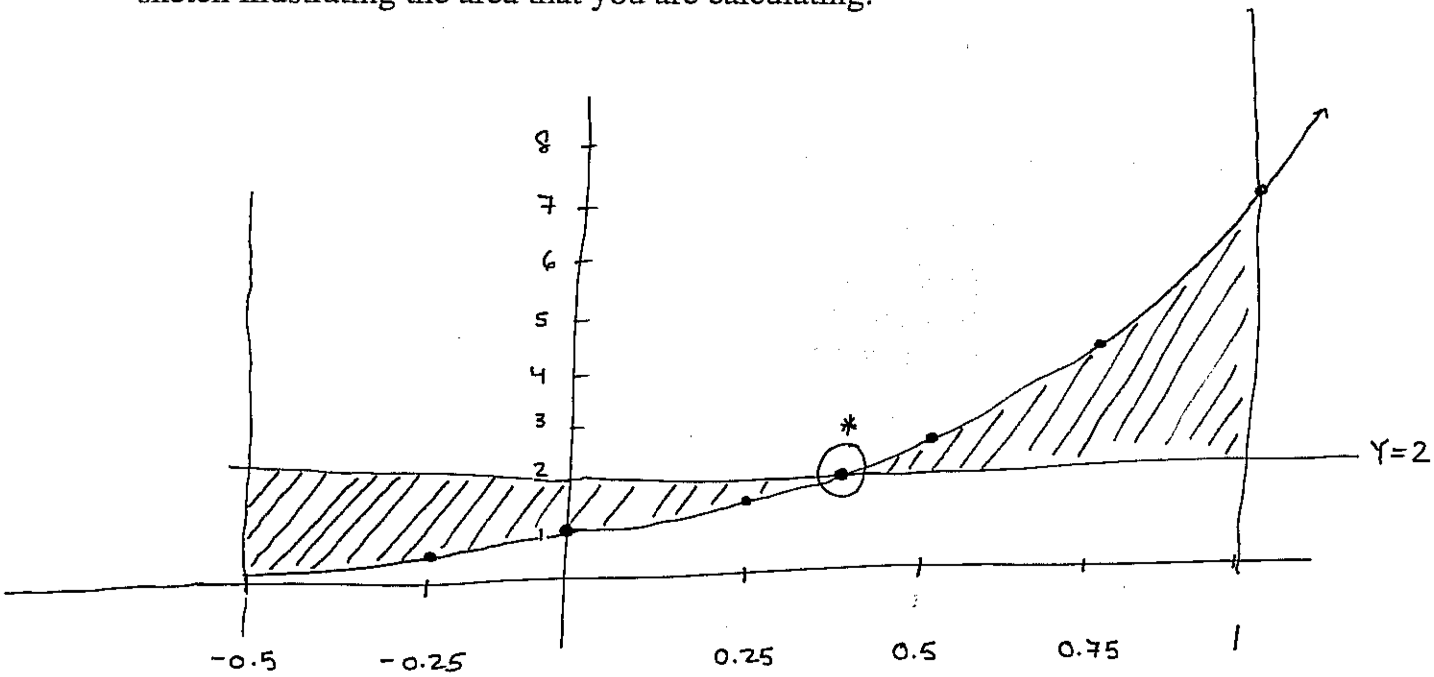
$1 = 0^2 + e^0 + \sin(0) - 6(0) + C$

$1 = 1 + C$

$C = 0$

so $\boxed{f(x) = x^2 + e^x + \sin x - 6x}$

Question 3. *(10 marks)*
Find the area bounded by the curves $y = e^{2x}$, $y = 2$, $x = -0.5$ and $x = 1$. Include a sketch illustrating the area that you are calculating.



* ⊙ INTERSECTION

$$Y = e^{2x} \qquad Y = 2$$

$$e^{2x} = 2$$
$$\ln e^{2x} = \ln 2$$
$$2x = \ln 2$$
$$x = \ln 2/2$$

$$\text{AREA} = \int_{-1/2}^{\ln 2/2} 2 - e^{2x}\,dx + \int_{\ln 2/2}^{1} e^{2x} - 2\,dx$$

$$= 2x - \tfrac{1}{2}e^{2x}\Big|_{-1/2}^{\ln 2/2} + \tfrac{1}{2}e^{2x} - 2x\Big|_{\ln 2/2}^{1}$$

$$= \left[(\ln 2 - 1) - (-1 - \tfrac{1}{2}e^{-1})\right] + \left[(\tfrac{1}{2}e^{2} - 2) - (1 - \ln 2)\right]$$

$$= \left(\ln 2 + \frac{1}{2e}\right) + \left(\frac{e^2}{2} - 3 + \ln 2\right)$$

$$= 0.877 + 1.387 \approx \boxed{2.265}$$

Question 4. *(10 marks)*
The trapezoid rule states that:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\frac{f(a)}{2}+f(x_1)+f(x_2)+\cdots+f(x_{n-1})+\frac{f(b)}{2}\right)$$

(a) Approximate the value of the integral $\int_0^1 20^x\,dx$.
Use the trapezoid rule with n = 5 intervals.

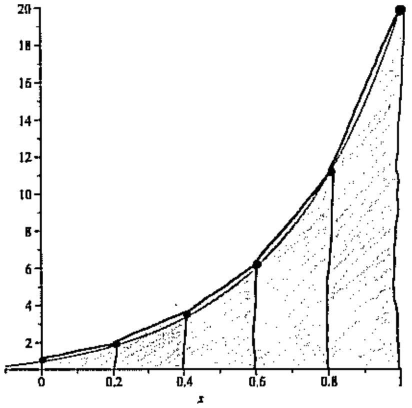
$b=1$, $a=0$, $n=5$ $\quad \frac{b-a}{n} = 0.2$

$f(a) = 20^0 = 1$
$f(x_1) = 20^{0.2} = 1.82$
$f(x_2) = 20^{0.4} = 3.31$
$f(x_3) = 20^{0.6} = 6.03$
$f(x_4) = 20^{0.8} = 10.986$
$f(b) = 20^1 = 20$

$$\int_0^1 20^x\,dx \approx 0.2\left(\frac{1}{2} + 1.82 + 3.31 + 6.03 + 10.986 + \frac{20}{2}\right)$$

$$= \boxed{6.531}$$

(b) Using the graph of $f(x)$ below, shade the area that you found by using the trapezoid rule with $n = 5$. Is the area larger or smaller than the value of the actual integral?



THE APPROXIMATION SHOULD GIVE A SLIGHTLY LARGER VALUE THAN THE ACTUAL INTEGRAL.

(c) Find the actual value of the integral.

$$\int_0^1 20^x\,dx = \frac{1}{\ln 20} 20^x \Big|_0^1 = \frac{1}{\ln 20}\left(20^1 - 20^0\right)$$

$$= \boxed{6.342}$$

Question 5. *(5 marks)*
The general expression for the slope of the tangent line to the curve $f(x)$ is $\frac{(\ln x)^2}{x}$. If the curve passes through the point $(1,2)$, find its expression.

$$f(x) = \int \frac{(\ln(x))^2}{x}\,dx \qquad u = \ln x, \quad du = \frac{1}{x}dx$$

$$= \int u^2\,du = u^3/3 + c = \frac{(\ln x)^3}{3} + c$$

SO $$2 = \frac{(\ln(1))^3}{3} + c$$

$$c = 2$$

$$\boxed{f(x) = \frac{1}{3}\ln^3 x + 2}$$

Question 6. *(5 marks)*
Integrate $\int (x^2)(x^3-2)^2\,dx$ in **two** different ways. Once with substitution and once without.

WITH substitution

$$u = x^3 - 2$$
$$du = 3x^2\,dx$$
$$\frac{1}{3}du = x^2\,dx$$

$$\int \frac{1}{3}u^2\,d$$
$$= \frac{1}{9}u^3 + c$$
$$\boxed{= \frac{1}{9}(x^3-2)^3 + c}$$

WITHOUT

$$\int x^2(x^3-2)^2$$
$$= \int x^2(x^6 - 4x^3 + 4)\,dx$$
$$= \int x^8 - 4x^5 + 4x^2\,dx$$
$$= \boxed{\frac{x^9}{9} - \frac{4x^6}{6} + \frac{4x^3}{3} + c}$$

Question 7. *(20 marks)*

Integrate each of the functions. *(5 marks each)*

(a) $\int \frac{(1+3e^{-2x})^4}{e^{2x}}\,dx$

$u = 1 + 3e^{-2x}$

$du = -6e^{-2x}\,dx$

$-\frac{1}{6}du = \frac{1}{e^{2x}}\,dx$

$= \int -\frac{1}{6} u^4\,du$

$= -\frac{u^5}{30} + c$

$= \boxed{-\frac{(1+3e^{-2x})^5}{30} + c}$

(b) $\int \frac{\tan^3 x}{\cos^2 x}\,dx$

$u = \tan x$

$du = \sec^2 x\,dx$

$du = \frac{1}{\cos^2 x}\,dx$

$= \int u^3\,du$

$= u^4/4 + c$

$= \boxed{\tan^4 x + c}$

(c) $\int 8\sin^{\frac{1}{3}} x \cos x \, dx$

$U = \sin x$
$du = \cos x \, dx$

$= \int 8 \; U^{1/3} \; du$

$= \quad 8\left(\frac{3}{4} U^{4/3}\right) + C$

$= \boxed{6 \sin^{4/3} x \;\; + C}$

(d) $\int \frac{1}{x \ln x} \, dx$

$U = \ln x$
$du = \frac{1}{x} \, dx$

$= \int \frac{1}{U} \, du$

$= \quad \ln|U| + C$

$= \boxed{\ln(\ln x) + C}$

Question 8. *(10 marks)*
Find the value of the following definite integrals. *(5 marks each)*

(a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{2\cos(2\theta)}{1+\sin^2(2\theta)}\,d\theta$

$u = \sin 2\theta$
$du = 2\cos 2\theta \, d\theta$

$\theta = \pi/2 \quad u = \sin \pi = 0$
$\theta = \pi/6 \quad u = \sin(\pi/3) = \sqrt{3}/2$

$$= \int_{\sqrt{3}/2}^{0} \frac{1}{1+u^2}\,du$$

$$= \arctan u \Big|_{\sqrt{3}/2}^{0}$$

$$= 0 - \arctan \sqrt{3}/2$$

$$= \boxed{-\arctan \sqrt{3}/2}$$

(b) $\displaystyle\int_{0}^{\ln 3} e^x\sqrt{e^x+1}\,dx$

$u = e^x + 1$
$du = e^x \, dx$
$x = \ln 3 \quad u = e^{\ln 3} + 1 = 4$
$x = 0 \quad u = e^0 + 1 = 2$

$$= \int_2^4 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2} \Big|_2^4$$

$$= \frac{2}{3}\left(4^{3/2} - 2^{3/2}\right)$$

$$= \frac{2}{3}\left(8 - 2\sqrt{2}\right)$$

$$\approx \boxed{3.45}$$