Dawson College: Calculus II: 201-NYB-05 05

February 18, 2011

Last Name: SOLUTIONS

First Name: ____________

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Test 1 (A)

Question 1.

(a) *(5 marks)* Evaluate the definate integral using Riemann sums.

$$\int_2^4 2-3x+x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$\Delta x = \frac{4-2}{n} = \frac{2}{n}$ $\quad x_i = 2+\frac{2i}{n}$

$$f(x_i) = 2-3\left(2+\frac{2i}{n}\right)+\left(2+\frac{2i}{n}\right)^2 = 2-6-\frac{6i}{n}+4+\frac{8i}{n}+\frac{4i^2}{n^2}$$

$$= \frac{2i}{n}+\frac{4i^2}{n^2}$$

$$f(x_i)\Delta x = \frac{4i}{n^2}+\frac{8i^2}{n^3}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{4i}{n^2}+\frac{8i^2}{n^3}\right) = \frac{4}{n^2}\sum_{i=1}^{n} i + \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{4}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\int_2^4 2-3x+x^2\,dx = \lim_{n\to\infty}\left[2\cdot\frac{n}{n}\left(1+\frac{1}{n}\right)+\frac{4}{3}\cdot\frac{n}{n}\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)\right]$$

$$= 2\cdot 1\cdot 1 + \frac{4}{3}\cdot 1\cdot 1\cdot 2$$

$$= \frac{14}{3}$$

(b) *(2 marks)* Write the above integral as a limit of Riemann sums using left endpoints as sample points. Do not evaluate the limit.

$x_{i-1} = 2+\frac{2(i-1)}{n}$

$$\int_2^4 2-3x+x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(2-3\left(2+\frac{2(i-1)}{n}\right)+\left(2+\frac{2(i-1)}{n}\right)^2\right)\cdot\frac{2}{n}$$

Question 2. Evaluate the following integrals.

(a) *(4 marks)*

$$\int (e^t + t^{-1} + \tan t - 12t^3)\, dt$$

$= e^t + \ln|t| - \ln|\cos t| - 3t^4 + c$

(b) *(4 marks)*

$$\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\, d\theta = \int_0^{\pi/4} \left(\frac{1}{\cos^2\theta} + 1\right) d\theta$$

$= \int_0^{\pi/4} (\sec^2\theta + 1)\, d\theta = \left[\tan\theta + \theta\right]_0^{\pi/4}$

$= \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - \left(\tan 0 + 0\right)$

$= 1 + \frac{\pi}{4} - 0 - 0$

$= \frac{4+\pi}{4}$

(c) *(4 marks)*

$$\int \frac{\cos x}{1+\sin^2 x}\,dx$$

LET $u = \sin x$

$du = \cos x \, dx$

$\therefore dx = \dfrac{du}{\cos x}$

$$= \int \frac{\cancel{\cos x}}{1+u^2}\,\frac{du}{\cancel{\cos x}}$$

$$= \int \frac{1}{1+u^2}\,du = \arctan u + c$$

$$= \arctan(\sin x) + c$$

(d) *(4 marks)*

$$\int x^2\sqrt{x+2}\,dx$$

LET $u = x+2$

$du = dx$

$\therefore x = u-2$

$$= \int (u-2)^2\sqrt{u}\,du$$

$$= \int (u^2-4u+4)\,u^{1/2}\,du$$

$$= \int \left(u^{5/2} - 4u^{3/2} + 4u^{1/2}\right)du$$

$$= \frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{8}{3}u^{3/2} + c$$

$$= \frac{2}{7}(x+2)^{7/2} - \frac{8}{5}(x+2)^{5/2} + \frac{8}{3}(x+2)^{3/2} + c$$

Question 3. *(5 marks)* The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics and engineering. Use the midpoint rule and three rectangles ($n = 3$) to approximate $\operatorname{erf}(4)$.

$$\operatorname{erf}(4) = \frac{2}{\sqrt{\pi}} \int_0^4 e^{-t^2}\,dt$$

$$n = 3 \Rightarrow \Delta x = \frac{4-0}{3} = \frac{4}{3}$$

$$x_0 = 0, \quad x_1 = \frac{4}{3}, \quad x_2 = \frac{8}{3}, \quad x_3 = 4$$

$$\Rightarrow \bar{x}_1 = \frac{x_0 + x_1}{2} = \frac{2}{3}, \quad \bar{x}_2 = \frac{x_1 + x_2}{2} = 2, \quad \bar{x}_3 = \frac{x_2 + x_3}{2} = \frac{10}{3}$$

$$\therefore \int_0^4 e^{-t^2}\,dt \approx e^{-(\frac{2}{3})^2} \cdot \frac{4}{3} + e^{-(2)^2} \cdot \frac{4}{3} + e^{-(\frac{10}{3})^2} \cdot \frac{4}{3}$$

$$\approx \frac{4}{3}\left[0.641180388 + 0.018315638 + 0.000014945\right]$$

$$= \frac{4}{3}\left[0.659510971\right]$$

$$= 0.879347961$$

$$\therefore \operatorname{erf}(4) = \frac{2}{\sqrt{\pi}} \int_0^4 e^{-t^2}\,dt \approx \frac{2}{\sqrt{\pi}}\left[0.879347961\right]$$

$$= \frac{1.758695924}{1.772453851} = 0.99223792$$

Question 4. *(5 marks)* Find the average value that $f(\theta) = \sec\theta\tan\theta$ takes on the interval $[0, \pi/4]$.

AVE VALUE OF f ON $[0, \pi/4]$ $= \dfrac{1}{\pi/4 - 0}\displaystyle\int_0^{\pi/4} f(x)\,dx$

$$= \frac{4}{\pi}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$$

$$= \frac{4}{\pi}\Big[\sec\theta\Big]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= \frac{4}{\pi}\left(\frac{1}{\frac{\sqrt{2}}{2}} - 1\right)$$

$$= \frac{4}{\pi}\left(\frac{2}{\sqrt{2}} - 1\right)$$

$$= \frac{8}{\pi\sqrt{2}} - \frac{4}{\pi}$$

Question 5. *(5 marks)* Compute the following derivative:

$$\frac{d}{dx}\left[\int_{\cos x}^{e^{x^2}} \arcsin(t+5)dt\right]$$

$$\int_{\cos x}^{e^{x^2}} \arcsin(t+5)dt = \int_{\cos x}^{0} \arcsin(t+5)dt + \int_{0}^{e^{x^2}} \arcsin(t+5)dt$$

$$= -\int_{0}^{\cos x} \arcsin(t+5)dt + \int_{0}^{e^{x^2}} \arcsin(t+5)dt$$

$$= -h_1(g_1(x)) + h_2(g_2(x))$$

BY FTC 2

WHERE

$h_1(x) = h_2(x) = \int_0^x \arcsin(t+5)dx \Rightarrow h_1'(x) = h_2'(x) = \arcsin(x+5)$

$g_1(x) = \cos x, \quad g_2(x) = e^{x^2} \Rightarrow g_1'(x) = -\sin x, \quad g_2'(x) = e^{x^2} \cdot 2x$

$$\frac{d}{dx}\left[\int_{\cos x}^{e^{x^2}} \arcsin(t+5)\,dt\right] = -h_1'(g_1(x)) \cdot g_1'(x) + h_2'(g_2(x)) \cdot g_2'(x)$$

$$= -\arcsin(\cos x+5)(-\sin x) + \arcsin(e^{x^2}+5) \cdot (2x e^{x^2})$$

$$= \sin x \cdot \arcsin(\cos x+5) + 2x e^{x^2} \cdot \arcsin(e^{x^2}+5)$$

Question 6. *(3 marks)*

If $F(x) = \int_1^x f(t)\,dt$ and $f(t) = \int_t^1 \sqrt{3u^2+5}\,du$

compute $F''(2)$.

$F'(x) = f(x) = \int_x^1 \sqrt{3u^2+5}\,du$ BY FTC2

$F''(x) = f'(x) = \frac{d}{dx}\left[\int_x^1 \sqrt{3u^2+5}\,du\right]$

$= \frac{d}{dx}\left[-\int_1^x \sqrt{3u^2+5}\,du\right]$

$= -\sqrt{3x^2+5}$

$\therefore F''(2) = -\sqrt{3(2)^2+5}$

$= -\sqrt{17}$

Bonus. *(3 marks)* Show directly that a definate integral

$$\int_a^b f(x)dx$$

computed using Riemann sums with left endpoints as sample points gives the same result as using right endpoints as sample points. (Hint: Think about what happens to the first and last approximating rectangles in the Riemann sums).

LEFT ENDPOINTS:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x = \lim_{n\to\infty}\left(f(x_0)\Delta x + \sum_{n=1}^{n-1} f(x_i)\Delta x \right)$$

$$= \lim_{n\to\infty}\left(f(x_0)\Delta x + \sum_{n=1}^{n-1} f(x_i)\Delta x + f(x_n)\Delta x - f(x_n)\Delta x \right)$$

$$= \lim_{n\to\infty}\left(f(x_0)\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x - f(x_n)\Delta x \right)$$

$$= \lim_{n\to\infty} f(x_0)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x - \lim_{n\to\infty} f(x_n)\Delta x$$

$$= 0 + \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x - 0 \qquad \left(\text{SINCE } \Delta x \to 0 \text{ AS } n \to \infty\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad \leftarrow \text{RIEMANN SUM USING RIGHT ENDPOINTS}$$