February 18, 2011

Last Name: SOLUTIONS

First Name: ______

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Test 1 (B)

Question 1.

(a) *(5 marks)* Evaluate the definate integral using Riemann sums.

$$\int_3^5 3-2x+x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{5-3}{n} = \frac{2}{n} \qquad x_i = 3+\frac{2i}{n}$$

$$f(x_i) = 3-2\left(3+\frac{2i}{n}\right)+\left(3+\frac{2i}{n}\right)^2 = 3-6-\frac{4i}{n}+9+\frac{12i}{n}+\frac{4i^2}{n^2}$$

$$= 6+\frac{8i}{n}+\frac{4i^2}{n^2}$$

$$f(x_i)\Delta x = \frac{12}{n}+\frac{16i}{n^2}+\frac{8i^2}{n^3}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n}\left(\frac{12}{n}+\frac{16i}{n^2}+\frac{8i^2}{n^3}\right) = \frac{12}{n}\sum_{i=1}^{n}1+\frac{16}{n^2}\sum_{i=1}^{n}i+\frac{8}{n^3}\sum_{i=1}^{n}i^2$$

$$= \frac{12}{n}\cdot n+\frac{16}{n^2}\frac{n(n+1)}{2}+\frac{8}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$\int_3^5 3-2x+x^2\,dx = \lim_{n\to\infty}\left[12+8\cdot\frac{n}{n}\left(1+\frac{1}{n}\right)+\frac{4}{3}\cdot\frac{n}{n}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 12+8\cdot1\cdot1+\frac{4}{3}\cdot1\cdot1\cdot2$$

$$= 12+8+\frac{8}{3}$$

$$= \frac{68}{3}$$

(b) *(2 marks)* Write the above integral as a limit of Riemann sums using left endpoints as sample points. Do not evaluate the limit.

$$x_{i-1} = 3+\frac{2(i-1)}{n}$$

$$\int_3^5 3-2x+x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(3-2\left(3+\frac{2(i-1)}{n}\right)+\left(3+\frac{2(i-1)}{n}\right)^2\right)\cdot\frac{2}{n}$$

Question 2. Evaluate the following integrals.
(a) *(3 marks)*

$$\int (x^{-1} + \tan x + e^x - 10x^4)\, dx$$

$$= \ln|x| - \ln|\cos x| + e^x - 2x^5 + C$$

(b) *(4 marks)*

$$\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\, d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2\theta} + 1\right) d\theta$$

$$= \int_0^{\pi/4} (\sec^2\theta + 1)\, d\theta$$

$$= \left[\tan\theta + \theta\right]_0^{\pi/4}$$

$$= \left(\tan\frac{\pi}{4} + \frac{\pi}{4}\right) - (\tan 0 + 0)$$

$$= 1 + \frac{\pi}{4} - 0 - 0$$

$$= \frac{4+\pi}{4}$$

(c) *(4 marks)*

$$\int x^2\sqrt{3+x}\,dx$$

LET $u = 3+x$

$du = dx$

$\Rightarrow \quad x = u-3$

$$= \int (u-3)^2 \sqrt{u}\,du$$

$$= \int (u^2 - 6u + 9)\sqrt{u}\,du$$

$$= \int u^{5/2} - 6u^{3/2} + 9u^{1/2}\,du = \frac{2}{7}u^{7/2} - \frac{12}{5}u^{5/2} + 6u^{3/2} + c$$

$$= \frac{2}{7}(3+x)^{7/2} - \frac{12}{5}(3+x)^{5/2} + 6(3+x)^{3/2} + c$$

(d) *(4 marks)*

$$\int \frac{\cos t}{1+\sin^2 t}\,dt$$

LET $u = \sin t$

$du = \cos t\,dt$

$dt = \dfrac{du}{\cos t}$

$$= \int \frac{\cos t}{1+u^2}\,\frac{du}{\cos t}$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \arctan u + C$$

$$= \arctan(\sin t) + c$$

Question 3. *(5 marks)* Find the average value that $f(\theta) = \sec\theta\tan\theta$ takes on the interval $[0, \pi/4]$.

AVE VALUE OF f ON $[0, \pi/4]$

$$= \frac{1}{\pi/4 - 0}\int_0^{\pi/4} f(x)\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/4} \sec\theta\tan\theta\, d\theta$$

$$= \frac{4}{\pi}\Big[\sec\theta\Big]_0^{\pi/4}$$

$$= \frac{4}{\pi}\left(\sec\frac{\pi}{4} - \sec 0\right)$$

$$= \frac{4}{\pi}\left(\frac{1}{\frac{\sqrt{2}}{2}} - 1\right)$$

$$= \frac{4}{\pi}\left(\frac{2}{\sqrt{2}} - 1\right)$$

$$= \frac{8}{\pi\sqrt{2}} - \frac{4}{\pi}$$

Question 4. *(5 marks)* The error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

is used in probability, statistics and engineering. Use the midpoint rule and three rectangles ($n = 3$) to approximate erf(4).

$$\operatorname{erf}(4) = \frac{2}{\sqrt{\pi}} \int_0^4 e^{-t^2}\,dt$$

$$n = 3 \Rightarrow \quad \Delta x = \frac{4-0}{3} = \frac{4}{3}$$

$$x_0 = 0, \quad x_1 = \frac{4}{3}, \quad x_2 = \frac{8}{3}, \quad x_3 = 4$$

$$\Rightarrow \bar{x}_1 = \frac{x_0 + x_1}{2} = \frac{2}{3}, \quad \bar{x}_2 = \frac{x_1 + x_2}{2} = 2, \quad \bar{x}_3 = \frac{x_2 + x_3}{2} = \frac{10}{3}$$

$$\therefore \int_0^4 e^{-t^2}\,dt \approx e^{-(2/3)^2} \cdot \frac{4}{3} + e^{-(2)^2} \cdot \frac{4}{3} + e^{-(\frac{10}{3})^2} \cdot \frac{4}{3}$$

$$\approx \frac{4}{3}\left[0.641180388 + 0.018315638 + 0.000014945\right]$$

$$= \frac{4}{3}\left[0.659510971\right]$$

$$= 0.0879347961$$

$$\therefore \operatorname{erf}(4) = \frac{2}{\sqrt{\pi}} \int_0^4 e^{-t^2}\,dt \approx \frac{2}{\sqrt{\pi}}\left[0.0879347961\right]$$

$$= 0.99223792$$

Question 5. *(5 marks)* Compute the following derivative:

$$\frac{d}{dx}\left[\int_{\sin x}^{\ln(x^2)} \arccos(t+2)dt\right]$$

$$\int_{\sin x}^{\ln x^2} \arccos(t+2)\,dt = \int_{\sin x}^{0} \arccos(t+2)\,dt + \int_{0}^{\ln x^2} \arccos(t+2)\,dt$$

$$= -\int_{0}^{\sin x} \arccos(t+2)\,dt + \int_{0}^{\ln x^2} \arccos(t+2)\,dt$$

$$= -h_1(g_1(x)) + h_2(g_2(x))$$

WHERE

$$h_1(x) = h_2(x) = \int_0^x \arccos(t+2)\,dt \Rightarrow h_1'(x) = h_2'(x) = \arccos(t+2)$$

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$$g_1(x) = \sin x;\ g_2(x) = \ln x^2 \Rightarrow g_1'(x) = \cos x;\ g_2'(x) = \frac{1}{x^2}\cdot 2x = \frac{2}{x}$$

$$\therefore \frac{d}{dx}\left[\int_{\sin x}^{\ln x^2} \arccos(t+2)\,dt\right] = -h_1'(g_1(x))\cdot g_1'(x) + h_2'(g_2(x))\cdot g_2'(x)$$

$$= -\arccos(\sin x+2)\cdot\cos x + \arccos(\ln x^2+2)\cdot\frac{2}{x}$$

Question 6. *(3 marks)*

If $F(x) = \int_1^x f(t)\,dt$ and $f(t) = \int_t^1 \sqrt{2u^2+4}\,du$

compute $F''(3)$.

$$F'(x) = f(x) = \int_x^1 \sqrt{2u^2+4}\,du$$

$$F''(x) = f'(x) = \frac{d}{dx}\left[\int_x^1 \sqrt{2u^2+4}\,du\right]$$

$$= \frac{d}{dx}\left[-\int_1^x \sqrt{2u^2+4}\,du\right]$$

$$= -\sqrt{2x^2+4}$$

$$\therefore \quad F''(3) = -\sqrt{2(3)^2+4}$$

$$= -\sqrt{22}$$

Bonus. *(3 marks)* Show directly that a definate integral

$$\int_a^b f(x)dx$$

computed using Riemann sums with left endpoints as sample points gives the same result as using right endpoints as sample points. (Hint: Think about what happens to the first and last approximating rectangles in the Riemann sums).