Dawson College: Calculus II: 201-NYB-05 05

March 25, 2011

Last Name: SOLUTIONS

First Name:

Student ID:

Test 2 (A)

Question 1. *(5 marks).* Evaluate the indefinite integral.

$$\int_0^1 \arctan x\, dx$$

LET $u = \arctan x \qquad dv = dx$

$du = \frac{1}{x^2+1}\,dx \qquad v = x$

$$= x\arctan x\Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx$$

LET $t = x^2+1$

$dt = 2x\,dx$

IF $x = 0 \Rightarrow t = 1$

$x = 1 \Rightarrow t = 2$

$$= x\arctan x\Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{t}\,dt$$

$$= x\arctan x\Big|_0^1 - \frac{1}{2}\ln|t|\Big|_1^2$$

$$= 1\cdot\arctan(1) - 0\cdot\arctan 0 - \frac{1}{2}\ln|2| + \frac{1}{2}\ln|1|$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Question 2. *(3 marks).* Evaluate the indefinite integral.

$$\int \sin^2 3x \, dx$$

$$= \int \frac{1 - \cos 6x}{2} \, dx$$

$$= \int \frac{1}{2} - \frac{1}{2} \cos 6x \, dx$$

$$= \frac{1}{2} x - \frac{1}{12} \sin 6x + C$$

Question 3. *(5 marks).*
(a) Prove the reduction formula

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx$$

LET $u = (\ln x)^n$ $\qquad dv = dx$

$du = n(\ln x)^{n-1} \cdot \frac{1}{x} dx \qquad v = x$

$\therefore \int (\ln x)^n dx = x(\ln x)^n - \int x n (\ln x)^{n-1} \cdot \frac{1}{x} dx$

$= x(\ln x)^n - n \int (\ln x)^{n-1} dx$

(b) Use this formula to evaluate

$$\int (\ln x)^3 dx$$

$= x(\ln x)^3 - 3\int (\ln x)^2 dx$

$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int (\ln x)^1 dx\right]$

$= x(\ln x)^3 - 3x(\ln x)^2 + 6\int \ln x \, dx$

$= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x \ln x - \int dx\right]$

$= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x + C$

Question 4. *(5 marks)* Evaluate the following integral.

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$$

LET $x = 3\sin\theta$ ON $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

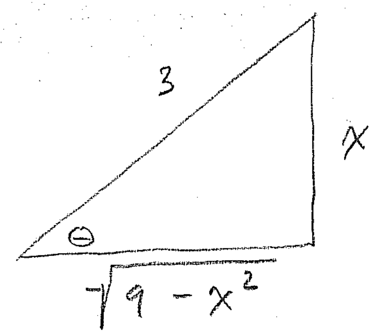
$dx = 3\cos\theta \, d\theta$

$$= \int \frac{1}{9\sin^2\theta \cdot 3\cos\theta} \cdot 3\cos\theta \, d\theta$$

$$= \frac{1}{9}\int \csc^2\theta \, d\theta$$

$$= -\frac{1}{9}\cot\theta + C$$

$$= -\frac{1}{9}\cdot\frac{\sqrt{9-x^2}}{x} + C$$



Question 5. *(3 marks)* Write the partial fraction decomposition of the following fraction. Do not determine the numerical values of the coefficients.

$$\frac{3x+9}{x^2(2x+1)^3(x^2+1)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+1} + \frac{D}{(2x+1)^2} + \frac{E}{(2x+1)^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

Question 6. *(5 marks)* Evaluate the following integral.

$$\int_{3\pi/4}^{\pi} \sec^6 x\,dx$$

LET $u = \tan x$

$du = \sec^2 x\,dx$

$$= \int_{\frac{3\pi}{4}}^{\pi} (1+\tan^2 x)^2 \sec^2 x\,dx$$

IF $x = \frac{3\pi}{4}$ $u = \tan \frac{3\pi}{4}$

$= -1$

$x = \pi$ $u = 0$

$$= \int_{-1}^{0} (1+u^2)^2\,du$$

$$= \int_{-1}^{0} 1 + 2u^2 + u^4\,dx$$

$$= \left[u + \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{-1}^{0}$$

$$= [0] - [-1 - \tfrac{2}{3} - \tfrac{1}{5}]$$

$$= \frac{28}{15}$$

Question 7. *(2 marks)* Evaluate the following integral.

$$\int_{-\pi/4}^{\pi/4} \frac{\sin x \cos x}{x^4+3x^2+1} dx$$

$$f(x) = \frac{\sin x \cos x}{x^4+3x^2+1}$$

$$f(-x) = \frac{\sin(-x)\cos(-x)}{(-x)^4+3(-x)^2+1}$$

$$= \frac{-\sin x \cos x}{x^4+3x^2+1} \leftarrow \sin x \text{ is ODD}$$

$$= -f(x)$$

$\therefore$ f is ODD

$$\therefore \int_{-\pi/4}^{\pi/4} \frac{\sin x \cos x}{x^4+3x^2+1} dx = 0$$

Question 8. *(5 marks)* Evaluate the following integral.

$$I = \int \frac{x^4+6x^2-20x+1}{x^2+2x+10}dx$$

$$\begin{array}{r l} & x^2 - 2x \\ x^2+2x+10 \, \big) & x^4+0x^3+6x^2-20x+1 \\ & -(x^4+2x^3+10x^2) \\ \hline & -2x^3-4x^2-20x \\ & -(-2x^3-4x^2-20x) \\ \hline & 0+1 \end{array}$$

$x^2+2x+10$

$= (x^2+2x+1)+10-1$

$= (x+1)^2+9$

$$\therefore I = \int x^2-2x + \frac{1}{x^2+2x+10}\,dx$$

$$= \int x^2-2x\,dx + \int \frac{1}{(x+1)^2+9}\,dx$$

$$= \int x^2-2x\,dx + \frac{1}{9}\int \frac{1}{\left(\frac{x+1}{3}\right)^2+1}\,dx$$

$$= \frac{1}{3}x^3 - x^2 + \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

Question 9. *(5 marks)* Evaluate the following integral.

$$I = \int \frac{x^2+x-18}{x^3+9x}dx = \int \frac{x^2+x-17}{x(x^2+9)} dx$$

$$\frac{x^2+x-18}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$x^2+x-17 = A(x^2+9) + (Bx+C)x$$

IF $x=0$

$-18 = 9A \Rightarrow \boxed{A = -2}$

IF $x = -1$

$-18 = -2(10) + B - C$

$-2 = B - C$

IF $x = 1$

$-16 = -2(10) + B + C$

$-16 + 20 = B + C$

$4 = B + C$

$\therefore 2 = 2B \Rightarrow \boxed{B = 3}$

$\therefore \boxed{C = 1}$

$$\therefore I = \int -\frac{2}{x} + \frac{3x+1}{x^2+9} dx$$

$$= \int -\frac{2}{x} dx + 3\int \frac{x}{x^2+9} dx + \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1} dx$$

$$= -2\ln|x| + \frac{3}{2}\ln(x^2+9) + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Question 10. *(3 marks)*. If f is continuous and

$$\int_1^3 f(x)dx = 36 \quad \text{find} \quad \int_0^1 x^2 f(2x^3+1)\,dx.$$

Clearly show your work.

LET $u = 2x^3 + 1$

$du = 6x^2\,dx \Rightarrow dx = \frac{du}{6x^2}$

IF $x = 0 \Rightarrow u = 1$

$x = 1 \Rightarrow u = 3$

$$\therefore \int_0^1 x^2 f(2x^3+1)dx = \int_1^3 x^2 f(u) \frac{du}{6x^2}$$

$$= \frac{1}{6}\int_1^3 f(u)\,du = \frac{1}{6}(36) = 6$$

Bonus. *(3 marks)* Find

$$I = \int \frac{1}{x^4+1}dx$$

given that

$$\frac{1}{x^4+1} = \frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{x}+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right]$$

$$= \frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right] + \frac{1}{4}\left[\frac{1}{(x+\frac{1}{\sqrt{2}})^2+1} + \frac{1}{(x-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\right]$$

$$\therefore I = \frac{\sqrt{2}}{8}\ln|x^2+\sqrt{2}x+1| + \frac{\sqrt{2}}{4}\left[\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x+1)\right] + C$$

$$- \frac{2}{8}\ln|x^2+\sqrt{2}x+1|$$