Dawson College: Calculus II: 201-NYB-05 05

March 25, 2011

Last Name: SOLUTIONS

First Name: ____________

Student ID: ____________

Test 2 (B)

Question 1. *(3 marks).* Evaluate the indefinite integral.

$$\int \cos^2 5x\, dx = \int \frac{1+\cos 10x}{2}\, dx$$

$$= \int \frac{1}{2} + \frac{1}{2}\cos 10x\, dx$$

$$= \frac{1}{2}x + \frac{1}{20}\sin 10x + C$$

Question 2. *(5 marks)*. Evaluate the indefinite integral.

$$\int_0^1 \arctan x \, dx =$$

LET $u = \arctan x$ $\quad dv = dx$

$du = \dfrac{1}{x^2+1}\,dx \quad v = x$

$$= x\arctan x \Big|_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx$$

LET $p = x^2+1$

$dp = 2x\,dx$

$dx = \dfrac{1}{2x}\,dp$

IF $x = 0 \Rightarrow p = 1$

$x = 1 \Rightarrow p = 2$

$$= x\arctan x \Big|_0^1 - \frac{1}{2}\int_1^2 \frac{1}{p}\,dp$$

$$= x\arctan x \Big|_0^1 - \frac{1}{2}\ln|p| \Big|_1^2$$

$$= (1)\arctan(1) - 0\cdot\arctan 0 - \frac{1}{2}\ln|2| + \frac{1}{2}\ln|1|$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Question 3. *(3 marks)* Write the partial fraction decomposition of the following fraction. Do not determine the numerical values of the coefficients.

$$\frac{5x-1}{x^2(5x+2)^3(x^2+3)^2}$$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{5x+2} + \frac{D}{(5x+2)^2} + \frac{E}{(5x+2)^3} + \frac{Fx+G}{x^2+3} + \frac{Hx+I}{(x^2+3)^2}$$

Question 4. *(5 marks)* Evaluate the following integral.

$$\int \frac{1}{x^2\sqrt{16-x^2}}\,dx$$

LET $x = 4\sin\theta$ WHERE $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$dx = 4\cos\theta\, d\theta$

$$\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta}$$
$$= 4\sqrt{1-\sin^2\theta}$$
$$= 4\sqrt{\cos^2\theta} = 4|\cos\theta|$$
$$= 4\cos\theta$$

(SINCE $\sin\theta \ge 0$ ON $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$)

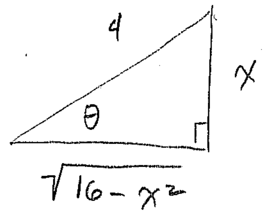
$$= \int \frac{4\cos\theta\, d\theta}{16\sin^2\theta \cdot 4\cos\theta}$$

$$= \frac{1}{16}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{16}\int \csc^2\theta\, d\theta$$

$$= -\frac{1}{16}\cot\theta + C$$

$$= -\frac{\sqrt{16-x^2}}{16x} + C$$



$$\cot\theta = \frac{\sqrt{16-x^2}}{x}$$

Question 5. *(5 marks).*
(a) Prove the reduction formula

$$\int (\ln x)^n dx = x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

LET $u = (\ln x)^n \quad dv = dx$

$du = n(\ln x)^{n-1} \cdot \frac{1}{x}\, dx \quad v = x$

$$\therefore \int (\ln x)^n dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1} \cdot \frac{1}{x}\, dx$$

$$= x(\ln x)^n - n\int (\ln x)^{n-1} dx$$

(b) Use this formula to evaluate

$$\int (\ln x)^3 dx = x(\ln x)^3 - 3\int (\ln x)^2 dx$$

$$= x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int (\ln x)^1 dx\right]$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\int (\ln x)dx$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6\left[x(\ln x) - \int dx\right]$$

$$= x(\ln x)^3 - 3x(\ln x)^2 + 6x(\ln x) - 6x + C$$

Question 6. *(2 marks)* Evaluate the following integral.

$$\int_{-\pi/3}^{\pi/3} \underbrace{\frac{(x^6 - x^2 + 2)\sin x}{\cos x}}_{\text{INTEGRAND IS ODD.}} dx$$

$$f(x) = \frac{(x^6 - x^2 + 2)\sin x}{\cos x}$$

$$f(-x) = \frac{\left[(-x)^6 - (-x)^2 + 2\right] \sin(-x)}{\cos(-x)}$$

$$= \frac{(x^6 - x^2 + 2)(-\sin x)}{\cos x} \leftarrow \sin x \text{ IS ODD.}$$

$$= -\frac{(x^6 - x^2 + 2)\sin x}{\cos x}$$

$$= -f(x) \quad \therefore f \text{ IS ODD}$$

$$\therefore \int_{-\pi/3}^{\pi/3} \frac{(x^6 - x^2 + 2)\sin x}{\cos x} dx = 0$$

Question 7. *(5 marks)* Evaluate the following integral.

$$\int_{3\pi/4}^{\pi} \sec^6 x\, dx$$

$$= \int_{\frac{3\pi}{4}}^{\pi} \sec^4 x \sec^2 x\, dx$$

$$= \int_{\frac{3\pi}{4}}^{\pi} (1+\tan^2 x)^2 \sec^2 x\, dx$$

$$= \int_{-1}^{0} (1+u^2)^2\, du$$

$$= \int_{-1}^{0} 1+2u^2+u^4\, du$$

$$= \left[u + \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{-1}^{0}$$

$$= [0] - \left[-1 - \frac{2}{3} - \frac{1}{5}\right]$$

$$= \frac{28}{15}$$

LET $u = \tan x$

$du = \sec^2 x\, dx$

IF $x = \frac{3\pi}{4} \Rightarrow u = -1$

$x = \pi \Rightarrow u = 0$

Question 8. *(5 marks)* Evaluate the following integral.

$$I = \int \frac{x^2+x-18}{x^3+9x}dx = \int \frac{x^2+x-18}{x(x^2+9)}\,dx$$

$$\frac{x^2+x-18}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}$$

$$x^2+x-18 = A(x^2+9) + (Bx+C)x$$

IF $x=0$

$-18 = 9A$

$\therefore -2 = A$

IF $x = 1$

$-16 = -2(10) + (B+C)$

① $4 = B+C$

IF $x=-1$

$-18 = -2(10) + B - C$

② $2 = B - C$

①+② :

$\therefore \; 6 = 2B \Rightarrow 3 = B$

$\therefore \; C = 1$

$$\therefore I = \int -\frac{2}{x} + \frac{3x+1}{x^2+9}\,dx$$

$$= -2\int \frac{1}{x}dx + 3\int \frac{x}{x^2+9}dx + \int \frac{1}{x^2+9}dx$$

$$= -2\ln|x| + \frac{3}{2}\ln|x^2+9| + \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}dx$$

$$= -2\ln|x| + \frac{3}{2}\ln|x^2+9| + \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

Question 9. *(5 marks)* Evaluate the following integral.

$$I = \int \frac{x^4+6x^2-20x+1}{x^2+2x+10}\,dx$$

$$\begin{array}{r} x^2 - 2x \\ x^2+2x+10 \,\overline{\big)\, x^4 + 0x^3 + 6x^2 - 20x + 1} \\ -\left(x^4 + 2x^3 + 10x^2\right) \\ \hline -2x^3 - 4x^2 - 20x \\ -\left(-2x^3 - 4x^2 - 20x\right) \\ \hline 0 + 1 \end{array}$$

$$\begin{aligned} & x^2+2x+10 \\ &= (x^2+2x+1) + 10 - 1 \\ &= (x+1)^2 + 9 \end{aligned}$$

$$\therefore\ I = \int x^2 - 2x + \frac{1}{x^2+2x+10}\,dx$$

$$= \int x^2 - 2x\,dx + \int \frac{1}{(x+1)^2+9}\,dx$$

$$= \frac{1}{3}x^3 - x^2 + \int \frac{1}{\left(\frac{x+1}{3}\right)^2 + 1}\,dx$$

$$= \frac{1}{3}x^2 - x^2 + \frac{1}{3}\arctan\left(\frac{x+1}{3}\right) + C$$

Question 10. *(3 marks)*. If f is continuous and

$$\int_1^4 f(x)\,dx = 24 \quad \text{find} \quad \int_0^1 x^3 f(3x^4+1)\,dx.$$

Clearly show your work.

LET $u = 3x^4+1$

$du = 12x^3\,dx \Rightarrow dx = \dfrac{du}{12x^3}$

$\therefore$ IF $x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=4$

$\therefore \displaystyle\int_0^1 x^3 f(3x^4+1)\,dx = \int_1^4 x^3 f(u)\,\frac{du}{12x^3}$

$\displaystyle = \frac{1}{12}\int_1^4 f(u)\,du = \frac{1}{12}(24) = 2$

Bonus. *(3 marks)* Find

$$I = \int \frac{1}{x^4+1}dx$$

given that

$$\frac{1}{x^4+1} = \frac{\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2+\sqrt{2}x+1} + \frac{-\frac{1}{2\sqrt{2}}x+\frac{1}{2}}{x^2-\sqrt{2}x+1}$$

$$= \frac{1}{4\sqrt{2}}\left[\frac{2x+2\sqrt{2}}{x^2+\sqrt{x}+1} - \frac{2x-2\sqrt{2}}{x^2-\sqrt{2}x+1}\right]$$

$$= \frac{\sqrt{2}}{8}\left[\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1} - \frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}\right] + \frac{1}{4}\left[\frac{1}{(x+\frac{1}{\sqrt{2}})^2+1} + \frac{1}{(x-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}\right]$$

$$\therefore I = \frac{\sqrt{2}}{8}\ln|x^2+\sqrt{2}x+1| + \frac{\sqrt{2}}{4}\left[\arctan(\sqrt{2}x+1) + \arctan(\sqrt{2}x+1)\right] + C$$

$$- \frac{2}{8}\ln|x^2+\sqrt{2}x+1|$$