Dawson College: Calculus II: 201-NYB-05 05 April 29, 2011

Test 3 (A)

Question 1. *(5 marks).* Evaluate the following improper integral.

$$\int_1^{\infty} (1-x)e^{-x}\,dx$$

Question 2. *(4 marks)* Sketch the region enclosed by the following curves. Write an expression that gives the area of this region (you do not need to evaluate).

$$y=x^2-4x+3,\quad y=-x^2+2x+3,\quad x=1,\quad \text{and} \quad x=4$$

Question 3. *(5 marks)* Find the arc length of the graph of

$$y=\sqrt{1-x^2}$$

on the interval $0 \le x \le \frac{1}{2}$.

Question 4. *(8 marks).* Set up an integral that gives the volume of the solid obtained by rotating the region bound by

$$y=\sqrt{x},\quad y=1,\quad x=4$$

about the line $x=5$ by
(a) Sketching a diagram and using the disk/washer method.
(b) Sketching a diagram and using the cylindrical shells method.

Question 5. *(4 marks)* Determine whether the following sequence converges or diverges. If it converges, find the limit.

$$a_n=\frac{\arctan(3n)}{n^2+3}$$

Question 6. *(10 marks)* Determine if each series converges or diverges. If it converges find the sum.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{4-3^n}{8^n}$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2}{4n^2-1}$ (Hint: Factor the denominator)

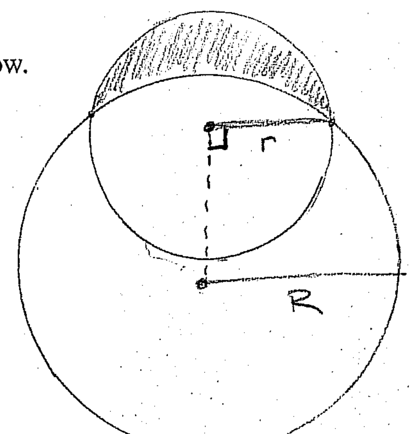
Question 7. *(12 marks)* Determine if each series converges or diverges.

(a) $\displaystyle\sum_{n=2}^{\infty}\frac{\sin^2 n}{\sqrt{n^3-1}}$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\ln(n^2)}$

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{5}{6^n-5}$

Bonus. *(3 marks)* Write an integral expressing the area of the shaded region below.



$$\int (1-x)e^{-x}\,dx \qquad \text{LET } u = 1-x \qquad dv = e^{-x}dx$$
$$du = -dx \qquad v = -e^{-x}$$

$$= uv - \int v\,du$$
$$= -(1-x)e^{-x} - \int e^{-x}dx$$
$$= xe^{-x} - e^{-x} + e^{-x} + C$$
$$= xe^{-x} + C$$

$$\therefore \int_1^\infty (1-x)e^{-x}\,dx = \lim_{t\to\infty}\int_1^t (1-x)e^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[xe^{-x}\right]_1^t$$
$$= \lim_{t\to\infty}\left[te^{-t} - e^{-1}\right]$$
$$= \lim_{t\to\infty} te^{-t} - \lim_{t\to\infty} e^{-1}$$
$$= \lim_{t\to\infty} te^{-t} - e^{-1} \qquad (\text{I.F. } \infty\cdot 0)$$
$$= \lim_{t\to\infty} \frac{t}{e^t} - e^{-1} \qquad \left(\text{I.F. } \frac{\infty}{\infty}\right)$$
$$\overset{H}{=} \lim_{t\to\infty} \frac{1}{e^t} - e^{-1}$$
$$= 0 - \frac{1}{e} = \boxed{-\frac{1}{e}}$$

2 INTERCEPTS!

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$2x^2 - 6x = 0$$

$$2x(x-3) = 0$$

$$\therefore x = 0,\ x = 3$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

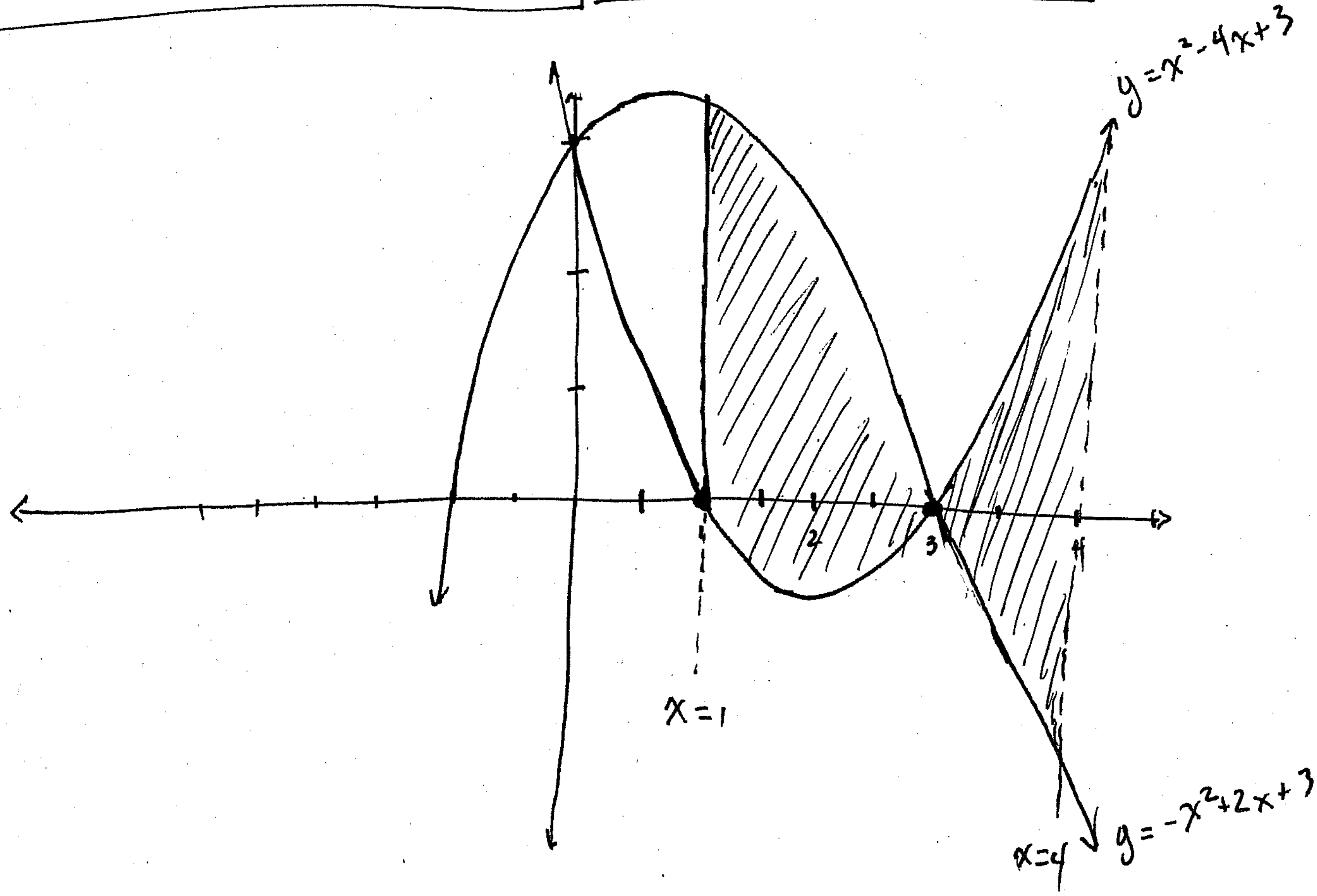
$$x = 3, 1$$

$$-x^2 + 2x + 3 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, -1$$



$$\therefore A = \int_1^3 \left[(-x^2 + 2x + 3) - (x^2 - 4x + 3)\right] dx + \int_3^4 \left[(x^2 - 4x + 3) - (-x^2 + 2x + 3)\right] dx$$

3

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

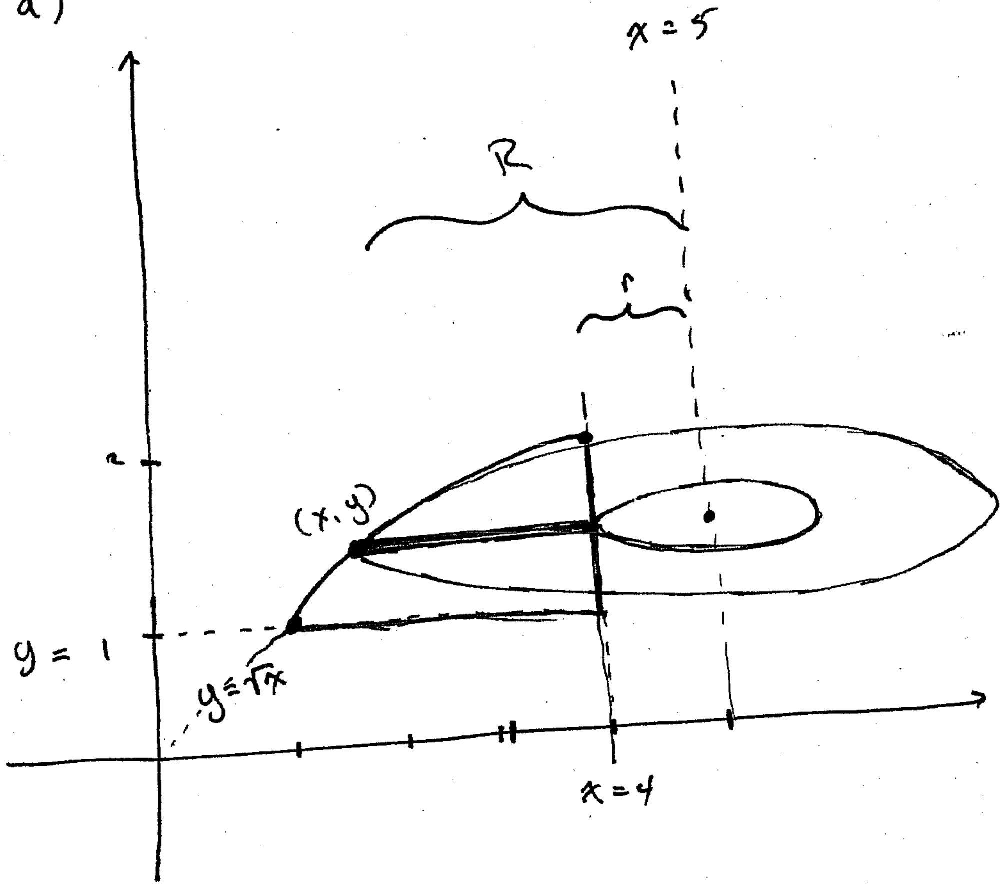
$$\therefore L = \int_0^{1/2} \sqrt{\frac{1}{1-x^2}}\,dx = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x \Big|_0^{1/2}$$

$$= \arcsin \frac{1}{2} - \arcsin 0$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

1) a)



$$A(y) = \pi R^2 - \pi r^2$$

$$R = 5 - x, \qquad r = 1$$
$$= 5 - y^2$$

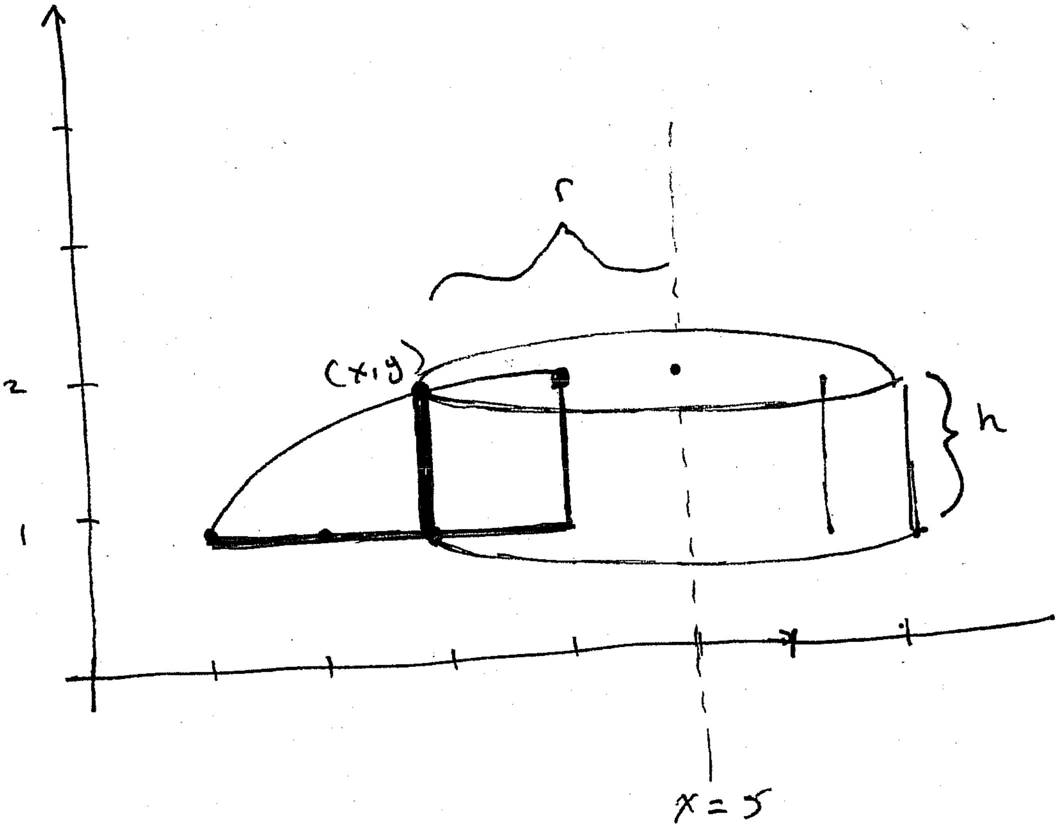
$$\therefore \quad V = \int_1^2 A(y)\,dy$$

$$= \int_1^2 \left[\pi(5-y^2)^2 - \pi(1)^2\right] dy$$

$$= \pi \int_1^2 (25 - 10y^2 + y^4 - 1)\,dy$$

$$= \pi \int_1^2 (24 - 10y^2 + y^4)\,dy$$

b)



$A(x) = 2\pi r h$

$r = 5 - x$

$h = y - 1$

$= \sqrt{x} - 1$

$\therefore \quad V = \int_1^4 2\pi (5-x)(\sqrt{x} - 1)\, dx$

5) $\quad -\frac{\pi}{2} \le \arctan(3n) \le \pi/2$ FOR ALL n.

$$\therefore \quad \frac{-\pi}{2(n^2+3)} \le \frac{\arctan(3n)}{n^2+3} \le \frac{\pi}{2(n^2+3)}$$

SINCE $\displaystyle\lim_{n\to\infty} \frac{-\pi}{2(n^2+3)} = 0$

AND $\displaystyle\lim_{n\to\infty} \frac{\pi}{2(n^2+3)} = 0$

$\displaystyle\lim_{n\to\infty} \frac{\arctan(3n)}{n^2+3} = 0$ BY SQUEEZE THM.

6) a) $$\sum_{n=2}^{\infty} \frac{4-3^n}{8^n} = \sum_{n=2}^{\infty}\left[\frac{4}{8^n} - \frac{3^n}{8^n}\right]$$

NOW

$$\sum_{n=2}^{\infty} \frac{4}{8^n} = \sum_{n=2}^{\infty} \frac{4}{8}\cdot\frac{1}{8^{n-1}} = \sum_{n=2}^{\infty} \frac{1}{2}\cdot\left(\frac{1}{8}\right)^{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{1}{2}\cdot\left(\frac{1}{8}\right)^{n-1} + \frac{1}{2}\left(\frac{1}{8}\right)^{0} - \frac{1}{2}\left(\frac{1}{8}\right)^{0}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{8}\right)^{n-1} - \frac{1}{2} \qquad \left(\begin{array}{c}\text{GEOMETRIC} \\ a=\frac{1}{2},\ r=\frac{1}{8}\end{array}\right)$$

$$= \frac{1/2}{1-\frac{1}{8}} - \frac{1}{2} = \frac{1}{2}\cdot\frac{8}{7} - \frac{1}{2} = \frac{1}{14}$$

AND

$$\sum_{n=2}^{\infty} \frac{3^n}{8^n} = \sum_{n=2}^{\infty}\left(\frac{3}{8}\right)^n = \sum_{n=2}^{\infty} \frac{3}{8}\left(\frac{3}{8}\right)^{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{3}{8}\left(\frac{3}{8}\right)^{n-1} + \frac{3}{8}\left(\frac{3}{8}\right)^{0} - \frac{3}{8}\left(\frac{3}{8}\right)^{0}$$

$$= \sum_{n=1}^{\infty} \frac{3}{8}\left(\frac{3}{8}\right)^{n-1} - \frac{3}{8} \qquad \left(\begin{array}{c}\text{GEOMETRIC} \\ a=\frac{3}{8},\ r=\frac{3}{8}\end{array}\right)$$

$$= \frac{3/8}{1-3/8} - \frac{3}{8} = \frac{3}{8}\cdot\frac{8}{5} - \frac{3}{8} = \frac{9}{40}$$

SINCE BOTH SERIES CONVERGE

$$\sum_{n=2}^{\infty}\left[\frac{4}{8^n}-\frac{3^n}{8^n}\right] = \sum_{n=2}^{\infty}\frac{4}{8^n} - \sum_{n=2}^{\infty}\frac{3^n}{8^n}$$

$$= \frac{1}{14} - \frac{9}{40}$$

$$= \frac{-43}{280}.$$

b) $$\sum_{n=1}^{\infty}\frac{2}{4n^2-1} = \sum_{n=1}^{\infty}\frac{2}{(2n+1)(2n-1)}$$

NOW

$$\frac{2}{(2n+1)(2n-1)} = \frac{A}{2n-1} + \frac{B}{2n+1}$$

$$\Rightarrow 2 = A(2n+1) + B(2n-1)$$

IF "$n=\frac{1}{2}$"

$$2 = 2A \Rightarrow A = 1$$

IF "$n=-\frac{1}{2}$"

$$2 = B(-2) \Rightarrow B = -1$$

$$\therefore \sum_{n=1}^{\infty}\frac{2}{4n^2-1} = \sum_{n=1}^{\infty}\frac{1}{2n-1} - \frac{1}{2n+1}$$

$$S_n = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{7}} - \cancel{\frac{1}{9}}\right)$$

$$+ \dots + \left(\cancel{\frac{1}{2n-5}} - \cancel{\frac{1}{2n-3}}\right) + \left(\cancel{\frac{1}{2n-3}} - \cancel{\frac{1}{2n-1}}\right) + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right)$$

$$= 1 - \frac{1}{2n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left[1 - \frac{1}{2n+1}\right] = 1 - 0 = 1$$

$\therefore$ THE SERIES CONVERGES AND

$$\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = 1$$

#7 a) $\dfrac{\sin^2 n}{\sqrt{n^3-1}} \leq \dfrac{1}{\sqrt{n^3-1}}$

IF $\sum \dfrac{1}{\sqrt{n^3-1}}$ CONVERGES THEN SO DOES $\sum \dfrac{\sin^2 n}{\sqrt{n^2-1}}$

BY COMPARISON TEST.

NOW, $\displaystyle \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^3-1}}}{\frac{1}{\sqrt{n^3}}} = \lim_{n\to\infty} \frac{\sqrt{n^3}}{\sqrt{n^3-1}} =$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\frac{n^3-1}{n^3}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^3}}} = \frac{1}{\sqrt{1-0}} = 1$$

SINCE $\displaystyle \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$ CONVERGES ($\cancel{P}$-SERIES, $p = 3/2$),

$\displaystyle \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$ CONVERGES BY LIMIT COMPARISON TEST.

$\therefore \displaystyle \sum_{n=2}^{\infty} \frac{\sin^2 n}{\sqrt{n^3-1}}$ CONVERGES.

b) NOTICE $f(x) = \frac{1}{x\ln(x^2)} \geq 0$ FOR $x \geq 2$

1) $f(x)$ IS DECREASING SINCE IF $x_2 > x_1$

WE HAVE THAT $\frac{1}{x_2 \ln(x_2^2)} \leq \frac{1}{x_1 \ln(x_1^2)}$

2) $\lim_{x\to\infty} \frac{1}{x\ln(x^2)} = 0$

NOW $\int \frac{1}{x\ln(x^2)}\,dx$

LET $u = \ln(x^2) = 2\ln x$

$du = 2 \cdot \frac{1}{x}\,dx$

$dx = \frac{x}{2}\,du$

$= \frac{1}{2}\int \frac{x}{xu}\,du$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln\left|\ln(x^2)\right| + C$

$\therefore \int_2^\infty \frac{1}{x\ln(x^2)}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln(x^2)}\,dx$

$= \lim_{t\to\infty} \frac{1}{2}\Big[\ln|\ln(x^2)|\Big]_2^t$

$= \lim_{t\to\infty} \frac{1}{2}\Big[\ln|\ln t^2| - \ln(\ln 4)\Big] = \infty$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n\ln(x^2)}$ DIVERGES BY INTEGRAL TEST.

c) LET $b_n = \dfrac{5}{6^n} = \dfrac{5}{6}\left(\dfrac{1}{6}\right)^{n-1}$

THEN $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\frac{5}{6^n-5}}{\frac{5}{6^n}} = \lim_{n\to\infty} \dfrac{6^n}{6^n-5}$

$$= \lim_{n\to\infty} \frac{1}{1-5/6^n} = 1$$

$\therefore$ SINCE $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \dfrac{5}{6}\left(\dfrac{1}{6}\right)^{n-1}$ CONVERGES

(GEOMETRIC, $r = \frac{1}{6}$)

$\sum_{n=1}^{\infty} \dfrac{5}{6^n-5}$ CONVERGES BY LIMIT COMPARISON TEST.