

## Test 3 (B)

**Question 1.** (4 marks) Sketch the region enclosed by the following curves. Write an expression that gives the area of this region (you do not need to evaluate).

$$y = x^2 - 4x + 3, \quad y = -x^2 + 2x + 3, \quad x = -1, \quad \text{and} \quad x = 2$$

**Question 2.** (5 marks). Evaluate the following improper integral.

$$\int_1^{\infty} (1-x)e^{-x} dx$$

**Question 3.** (8 marks). Set up an integral that gives the volume of the solid obtained by rotating the region bound by

$$y = \sqrt{x}, \quad y = 1, \quad x = 4$$

about the line  $x = 5$  by

(a) Sketching a diagram and using the disk/washer method.

(b) Sketching a diagram and using the cylindrical shells method.

**Question 4.** (5 marks) Find the arc length of the graph of

$$y = \sqrt{1-x^2}$$

on the interval  $0 \leq x \leq \frac{1}{2}$ .

**Question 5.** (4 marks) Determine whether the following sequence converges or diverges.

$$a_n = \frac{\arctan(3n)}{n^2 + 3}$$

**Question 6.** (10 marks) Determine if each series converges or diverges. If it converges find the sum.

(a)  $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$  (Hint: Factor the denominator)

(b)  $\sum_{n=2}^{\infty} \frac{4 - 3^n}{8^n}$

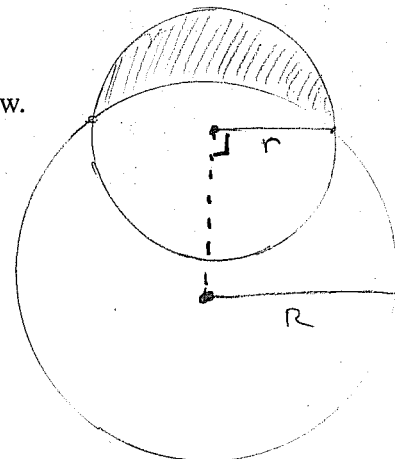
**Question 7.** (12 marks) Determine if each series converges or diverges.

(a)  $\sum_{n=1}^{\infty} \frac{5}{6^n - 5}$

(b)  $\sum_{n=2}^{\infty} \frac{\sin^2 n}{\sqrt{n^3 - 1}}$

(c)  $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^2)}$

**Bonus.** (3 marks) Write an integral expressing the area of the shaded region below.



#1

INTERCEPTS!

$$x^2 - 4x + 3 = -x^2 + 2x + 3$$

$$2x^2 - 6x = 0$$

$$2x(x-3) = 0$$

$$\therefore x=0, x=3$$

$$x^2 - 4x + 3 = 0$$

$$(x-3)(x-1) = 0$$

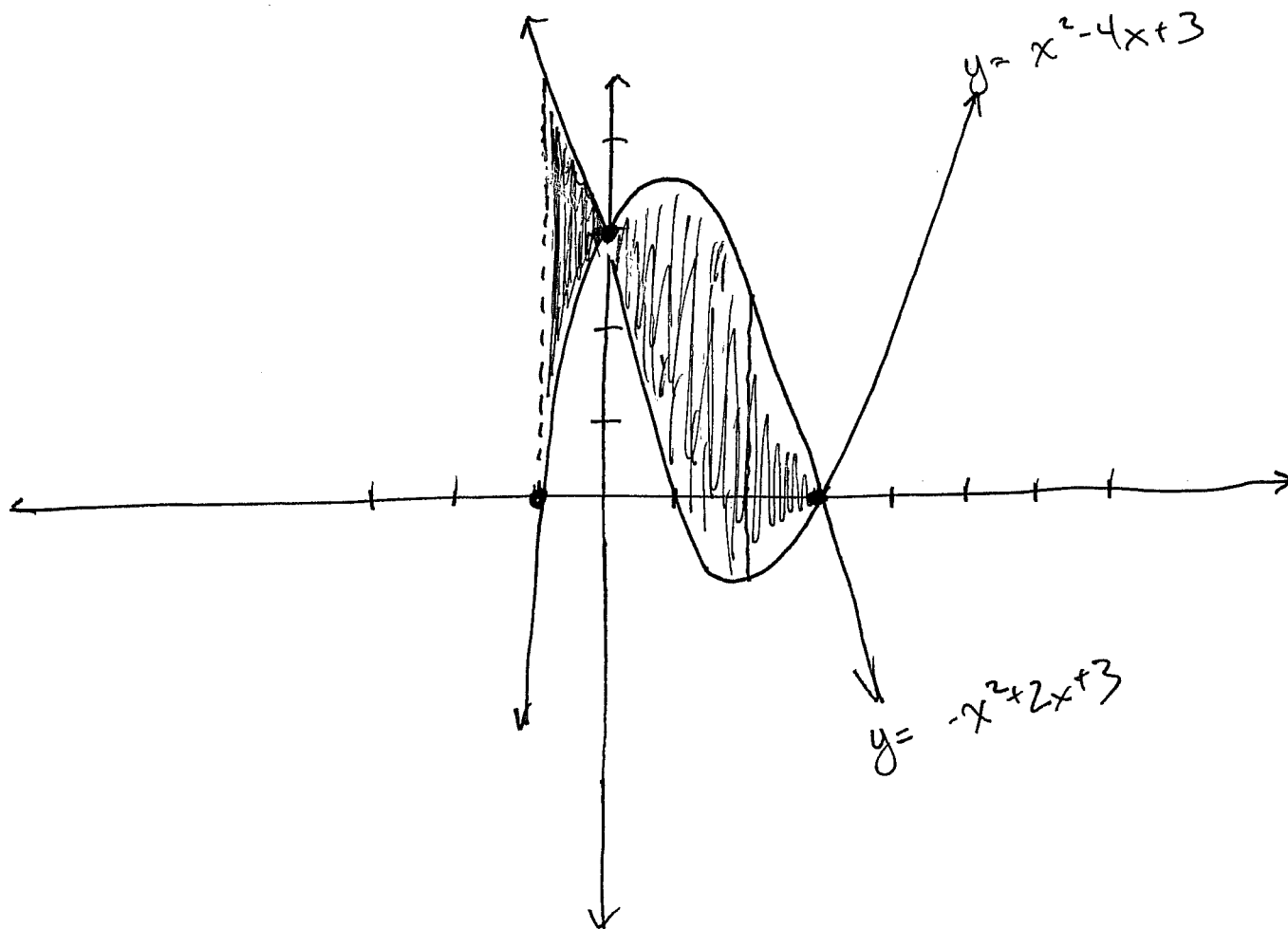
$$x = 3, 1$$

$$-x^2 + 2x + 3 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, -1$$



$$\therefore A = \int_{-1}^0 [(-x^2 + 2x + 3) - (x^2 - 4x + 3)] dx$$

$$+ \int_0^3 [(x^2 - 4x + 3) - (-x^2 + 2x + 3)] dx$$

$$2 \int (1-x)e^{-x} dx$$

$$\text{let } u = 1-x$$

$$dv = e^{-x} dx$$

$$du = -dx$$

$$v = -e^{-x}$$

$$= uv - \int v du$$

$$= -(1-x)e^{-x} - \int e^{-x} dx$$

$$= xe^{-x} - e^{-x} + e^{-x} + C$$

$$= xe^{-x} + C$$

$$\therefore \int_1^{\infty} (1-x)e^{-x} dx = \lim_{t \rightarrow \infty} \int_1^t (1-x)e^{-x} dx$$

$$= \lim_{t \rightarrow \infty} [xe^{-x}]_1^t$$

$$= \lim_{t \rightarrow \infty} [te^{-t} - e^{-1}]$$

$$= \lim_{t \rightarrow \infty} te^{-t} - \lim_{t \rightarrow \infty} e^{-1}$$

$$= \lim_{t \rightarrow \infty} te^{-t} - e^{-1}$$

(I.F.  $\infty \cdot 0$ )

$$= \lim_{t \rightarrow \infty} \frac{t}{e^t} - e^{-1}$$

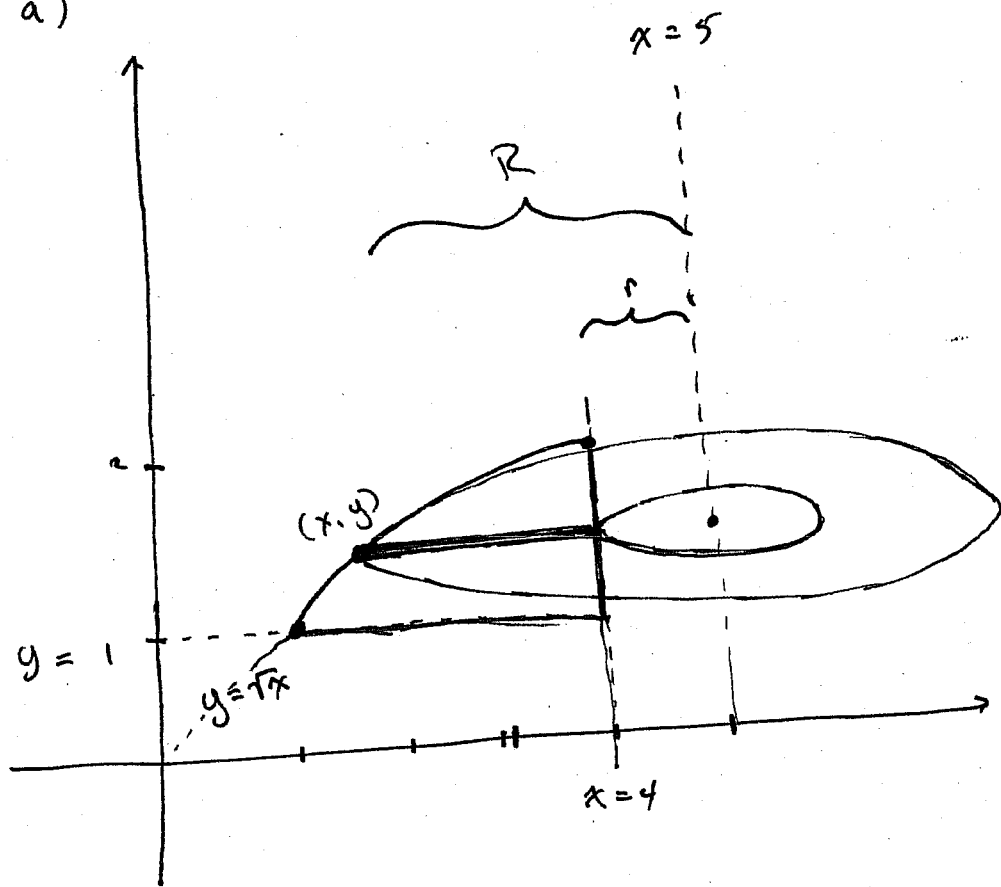
(I.F.  $\frac{\infty}{\infty}$ )

$$\stackrel{\textcircled{H}}{=} \lim_{t \rightarrow \infty} \frac{1}{e^t} - e^{-1}$$

$$= 0 - \frac{1}{e} =$$

$$\boxed{-\frac{1}{e}}$$

3 a)



$$A(y) = \pi R^2 - \pi r^2$$

$$R = 5 - x, \quad r = 1$$

$$= 5 - y^2$$

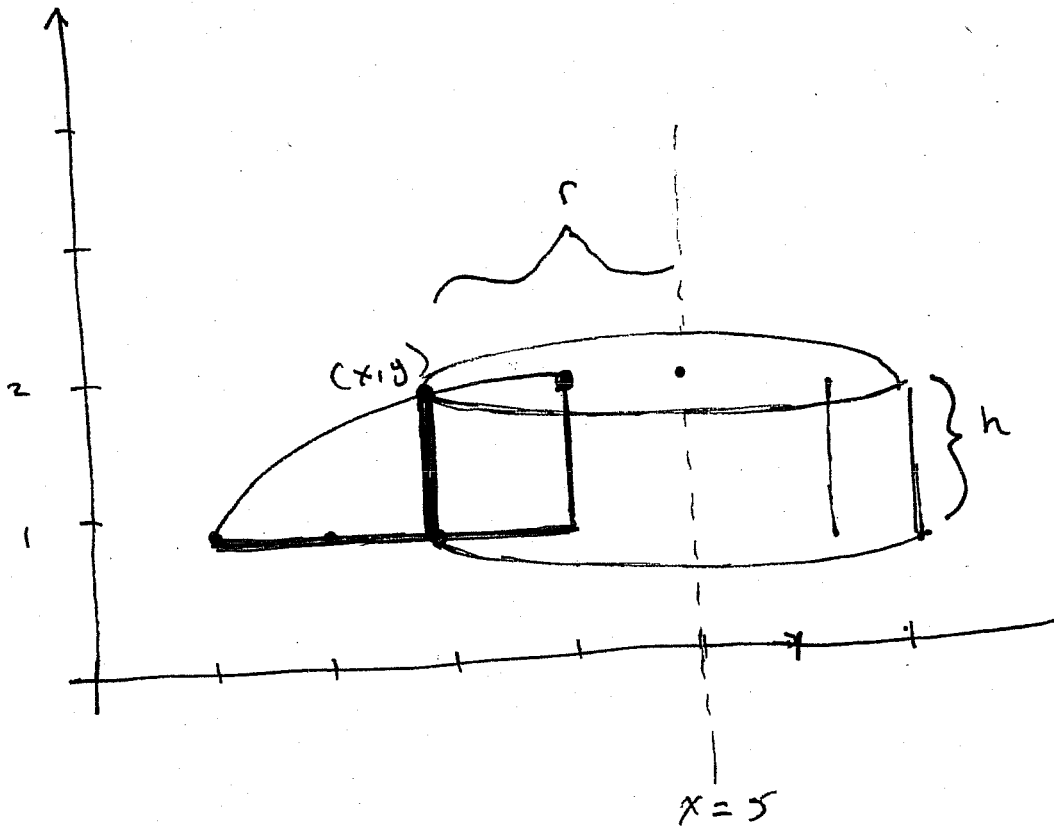
$$\therefore V = \int_1^2 A(y) dy$$

$$= \int_1^2 [\pi(5-y^2)^2 - \pi(1)^2] dy$$

$$= \pi \int_1^2 (25 - 10y^2 + y^4 - 1) dy$$

$$= \pi \int_1^2 (24 - 10y^2 + y^4) dy$$

b)



$$A(x) = 2\pi r h$$

$$r = 5 - x$$

$$h = y - 1$$

$$= \sqrt{x} - 1$$

$$\therefore V = \int_0^4 2\pi (5-x)(\sqrt{x}-1) dx$$

$$\frac{dy}{dx} = \frac{1}{2} (1-x^2)^{-1/2} (-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{1-x^2}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$\therefore L = \int_0^{1/2} \sqrt{\frac{1}{1-x^2}} dx = \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \Big|_0^{1/2}$$

$$= \arcsin \frac{1}{2} - \arcsin 0$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

$$5) \quad -\frac{\pi}{2} \leq \arctan(3n) \leq \frac{\pi}{2} \quad \text{FOR ALL } n.$$

$$\therefore \frac{-\pi}{2(n^2+3)} \leq \frac{\arctan(3n)}{n^2+3} \leq \frac{\pi}{2(n^2+3)}$$

$$\text{SINCE } \lim_{n \rightarrow \infty} \frac{-\pi}{2(n^2+3)} = 0$$

$$\text{AND } \lim_{n \rightarrow \infty} \frac{\pi}{2(n^2+3)} = 0$$

$$\lim_{n \rightarrow \infty} \frac{\arctan(3n)}{n^2+3} = 0 \quad \text{BY SQUEEZE THM.}$$

$$6) b) \sum_{n=2}^{\infty} \frac{4 - 3^n}{8^n} = \sum_{n=2}^{\infty} \left[ \frac{4}{8^n} - \frac{3^n}{8^n} \right]$$

NOW

$$\sum_{n=2}^{\infty} \frac{4}{8^n} = \sum_{n=2}^{\infty} \frac{4}{8} \cdot \frac{1}{8^{n-1}} = \sum_{n=2}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{8}\right)^{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{1}{2} \cdot \left(\frac{1}{8}\right)^{n-1} + \frac{1}{2} \left(\frac{1}{8}\right)^0 - \frac{1}{2} \left(\frac{1}{8}\right)^0$$

$$= \sum_{n=2}^{\infty} \frac{1}{2} \left(\frac{1}{8}\right)^{n-1} - \frac{1}{2} \quad \left( \text{GEOMETRIC} \right. \\ \left. a = \frac{1}{2}, r = \frac{1}{8} \right)$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{8}} - \frac{1}{2} = \frac{1}{2} \cdot \frac{8}{7} - \frac{1}{2} = \frac{1}{14}$$

AND

$$\sum_{n=2}^{\infty} \frac{3^n}{8^n} = \sum_{n=2}^{\infty} \frac{3}{8} \left(\frac{3}{8}\right)^{n-1}$$

$$= \sum_{n=2}^{\infty} \frac{3}{8} \left(\frac{3}{8}\right)^{n-1} + \frac{3}{8} \left(\frac{3}{8}\right)^0 - \frac{3}{8} \left(\frac{3}{8}\right)^0$$

$$= \sum_{n=2}^{\infty} \frac{3}{8} \left(\frac{3}{8}\right)^{n-1} - \frac{3}{8} \quad \left( \text{GEOMETRIC} \right. \\ \left. a = \frac{3}{8}, r = \frac{3}{8} \right)$$

$$= \frac{\frac{3}{8}}{1 - \frac{3}{8}} - \frac{3}{8} = \frac{3}{8} \cdot \frac{8}{5} - \frac{3}{8} = \frac{9}{40}$$



SINCE BOTH SERIES CONVERGE

$$\begin{aligned}\sum_{n=2}^{\infty} \left[ \frac{4}{8^n} - \frac{3^n}{8^n} \right] &= \sum_{n=2}^{\infty} \frac{4}{8^n} - \sum_{n=2}^{\infty} \frac{3^n}{8^n} \\ &= \frac{1}{14} - \frac{9}{40} \\ &= \frac{-43}{280}\end{aligned}$$

$$a) \sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n-1)}$$

now

$$\frac{2}{(2n+1)(2n-1)} = \frac{A}{2n+1} + \frac{B}{2n-1}$$

$$\Rightarrow 2 = A(2n+1) + B(2n-1)$$

$$\text{IF " } n = \frac{1}{2} \text{ "}$$

$$2 = 2A \Rightarrow A = 1$$

$$\text{IF " } n = -\frac{1}{2} \text{ "}$$

$$2 = B(-2) \Rightarrow B = -1$$

$$\therefore \sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \sum_{n=1}^{\infty} \frac{1}{2n+1} - \frac{1}{2n-1}$$

$$\begin{aligned}
 S_n &= \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) \\
 &+ \dots + \left(\frac{1}{2n-5} - \frac{1}{2n-3}\right) + \left(\frac{1}{2n-3} - \frac{1}{2n-1}\right) + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\
 &= 1 - \frac{1}{2n+1}
 \end{aligned}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left[ 1 - \frac{1}{2n+1} \right] = 1 - 0 = 1$$

∴ THE SERIES CONVERGES AND

$$\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = 1$$

$$7 a) \text{ LET } b_n = \frac{5}{6^n} = \frac{5}{6} \left(\frac{1}{6}\right)^{n-1}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\frac{5}{6^n - 5}}{\frac{5}{6^n}} = \lim_{n \rightarrow \infty} \frac{6^n}{6^n - 5}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 - 5/6^n} = 1$$

$\therefore$  SINCE  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{6} \left(\frac{1}{6}\right)^{n-1}$  CONVERGES

(GEOMETRIC,  $r = \frac{1}{6}$ )

$\sum_{n=1}^{\infty} \frac{5}{6^n - 5}$  CONVERGES BY LIMIT COMPARISON TEST.

$$b) \frac{\sin^2 n}{\sqrt{n^3-1}} \leq \frac{1}{\sqrt{n^3-1}}$$

IF  $\sum \frac{1}{\sqrt{n^3-1}}$  CONVERGES THEN SO DOES  $\sum \frac{\sin^2 n}{\sqrt{n^3-1}}$

BY COMPARISON TEST.

now,

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{\sqrt{n^3-1}}}{\frac{1}{\sqrt{n^3}}} = \lim_{n \rightarrow \infty} \frac{\sqrt{n^3}}{\sqrt{n^3-1}} =$$

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{\frac{n^3-1}{n^3}}} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1-\frac{1}{n^3}}} = \frac{1}{\sqrt{1-0}} = 1$$

SINCE  $\sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$  CONVERGES (P-SERIES,  $p = 3/2$ ),

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3-1}}$  CONVERGES BY LIMIT COMPARISON TEST.

$\therefore \sum_{n=2}^{\infty} \frac{\sin^2 n}{\sqrt{n^3-1}}$  CONVERGES.

c) NOTICE  $f(x) = \frac{1}{x \ln(x^2)} \geq 0$  FOR  $x \geq 2$

1)  $f(x)$  IS DECREASING SINCE IF  $x_2 > x_1$

WE HAVE THAT  $\frac{1}{x_2 \ln(x_2^2)} \leq \frac{1}{x_1 \ln(x_1^2)}$

2)  $\lim_{x \rightarrow \infty} \frac{1}{x \ln(x^2)} = 0$

Now  $\int \frac{1}{x \ln(x^2)} dx$

LET  $u = \ln(x^2) = 2 \ln x$

$du = 2 \cdot \frac{1}{x} dx$

$dx = \frac{x}{2} du$

$= \frac{1}{2} \int \frac{x}{x u} du$

$= \frac{1}{2} \int \frac{1}{u} du$

$= \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|\ln(x^2)| + C$

$\therefore \int_2^{\infty} \frac{1}{x \ln(x^2)} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln(x^2)} dx$

$= \lim_{t \rightarrow \infty} \frac{1}{2} \left[ \ln|\ln(x^2)| \right]_2^t$

$= \lim_{t \rightarrow \infty} \frac{1}{2} \left[ \ln|\ln t^2| - \ln(\ln 4) \right] = \infty$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n \ln(x^2)}$  DIVERGES BY INTEGRAL TEST.