EXAMPLE: FIND

$$\int \frac{x-3}{(x^2+2x+10)^2}\,dx = I$$

SOLUTION

$$\int \frac{x-3}{(x^2+2x+10)^2} = \int \frac{x+1-1-3}{(x^2+2x+10)^2}\,dx = \int \underbrace{\frac{x+1}{(x^2+2x+10)^2}}_{I_1} - \underbrace{\frac{4}{(x^2+2x+10)^2}}_{I_2}\,dx$$

$$I_1 = \int \frac{x+1}{(x^2+2x+10)^2}\,dx$$

LET $u = x^2+2x+10$

$du = (2x+2)\,dx$

$\therefore dx = \dfrac{du}{2(x+1)}$

$$= \frac{1}{2}\int \frac{du}{u^2}$$

$$= -\frac{1}{2}u^{-1} + C = -\frac{1}{2}\cdot\frac{1}{x^2+2x+10} + C$$

I_2:

$$x^2+2x+10 = (x^2+2x+1)+10-1 = (x+1)^2+9$$

$$= 9\left(\frac{x+1}{3}\right)^2 + 1$$

$$\therefore I_2 = \frac{4}{81}\int \frac{1}{\left[\left(\frac{x}{3}+\frac{1}{3}\right)^2+1\right]^2}\,dx$$

LET $u = \dfrac{x}{3}+\dfrac{1}{3}$

$du = \dfrac{1}{3}\,dx$

$dx = 3\,du$

$$= \frac{12}{81}\int \frac{1}{(u^2+1)^2}\,du$$

LET $u = \tan\theta$

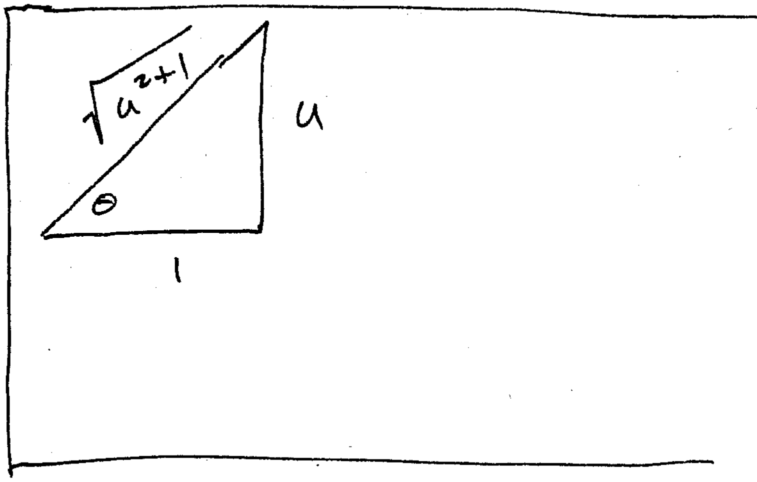
$du = \sec^2\theta\,d\theta$

$$= \frac{4}{27}\int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta$$

$$= \frac{4}{27}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta$$

$$= \frac{4}{27}\int \frac{1}{\sec^2\theta}\,d\theta = \frac{4}{27}\int \cos^2\theta\,d\theta$$

$$= \frac{4}{27}\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{2}{27}\int 1+\cos 2\theta\,d\theta$$



$$= \frac{2}{27}\left[\theta + \frac{\sin 2\theta}{2}\right] + C$$

$$= \frac{2}{27}\theta + \frac{\sin 2\theta}{27} + C$$

$$= \frac{2}{27}\arctan u + \frac{2\sin\theta\cos\theta}{27} + C$$

$$= \frac{2}{27}\arctan u + \frac{2}{27}\cdot\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} + C$$

$$= \frac{2}{27}\arctan u + \frac{2}{27}\cdot\frac{u}{u^2+1} + C$$

$$= \frac{2}{27}\arctan\left(\frac{x+1}{3}\right) + \frac{2}{27}\cdot\frac{\frac{x+1}{3}}{\left(\frac{x+1}{3}\right)^2+1} + C$$

$$\therefore I = -\frac{1}{2}\cdot\frac{1}{x^2+2x+10} + \frac{2}{27}\arctan\left(\frac{x+1}{3}\right)$$

$$+ \frac{2}{27}\cdot\frac{3(x+1)}{(x+1)^2+9} + C$$