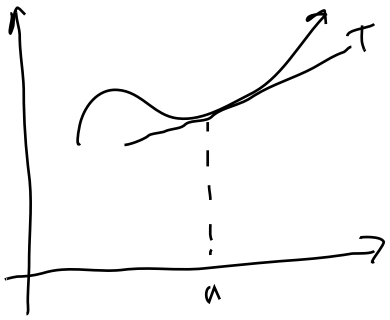
Calculus II

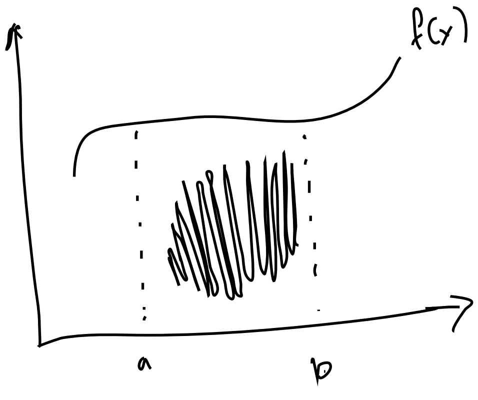
Recall that calculus was developed by Isaac Newton and Gottfried Wilhelm Leibniz in order to solve two main problems.

The first problem was investigated in calculus I: Find the tangent line to a curve at a given point on the curve.



In calculus I we found that the slope of the tangent line T to the curve at $x = a$ is given by $f'(a)$, the derivative at a.

The second problem will be investigated in calculus II: Find the area of a planar region bounded by a curve.



Like how the tangent line problem is related to the derivative of a function, we will see that the area problem is related to the antiderivative of a function.

Antiderivatives and the rules of Integration

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

$$F(x) \xrightarrow[\text{DERIVATIVE}]{} f(x) \xrightarrow[\text{DERIVATIVE}]{} f'(x)$$

$$F(x) \xleftarrow[\text{ANTIDERIVATIVE}]{} f(x) \xleftarrow[\text{ANTIDERIVATIVE}]{} f'(x)$$

Example: Show that $F(x) = \frac{1}{3}x^3 + 2x^2 - 5$ is an antiderivative of $f(x) = x^2 + 4x$.
Solution:

$F'(x) = x^2 + 4x = f(x)$

$\therefore$ F(x) IS THE ANTIDERIVATIVE OF f(x).

Notice that

$$\begin{aligned} F(x) &= \tfrac{1}{3}x^3 + 2x^2 - 5 \\ H(x) &= \tfrac{1}{3}x^3 + 2x^2 + 6 \\ P(x) &= \tfrac{1}{3}x^3 + 2x^2 \end{aligned}$$

are all antiderivatives of $f(x)$.

In fact, if $G(x)$ is any antiderivative of a function $f(x)$ then any antiderivative $F(x)$ of $f(x)$ must be of the form $F(x) = G(x) + C$ where C is a constant.

The operation of finding all antiderivatives is called antidifferentiation (or indefinite integration) and is denoted by the integral sign $\int$

If $F(x)$ is an antiderivative of $f(x)$ then

$$\int f(x)dx = F(x) + C$$

CONSTANT OF INTEGRATION

INTEGRAND

VARIABLE OF INTEGRATION

Note: The $\int$ and the dx are paired together. The notation requires **both**.

Example:

$$\int 5x^4dx = x^5 + C \qquad \text{SINCE } \frac{d}{dx}\left[x^5 + C\right] = 5x^4$$

Basic Rules of Integration

1. $\int kdx = kx + C$ SINCE $\frac{d}{dx}\left[kx + C\right] = k$

Ex: $\int 3dx = 3x + C$

2. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ SINCE $\frac{d}{dx}\left[\frac{x^{n+1}}{n+1} + C\right] = \frac{(n+1)x^n}{n+1} = x^n$

$(n \neq -1)$

Ex: $\int x^3 dx = \frac{x^4}{4} + C$

3. $\int cf(x)dx = c\int f(x)dx$

Ex: $\int 9x^3 dx = 9\int x^3 dx = 9\left[\frac{x^4}{4} + C\right] = \frac{9}{4}x^4 + 9C$

$= \frac{9}{4}x^4 + C_2$

4. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Ex: $\int \left(3x^5 - 4x^{3/2} + 2x^{-3}\right) dx = \int 3x^5 dx - \int 4x^{3/2} dx + \int 2x^{-3} dx$

$= \frac{3x^6}{6} - \frac{4x^{5/2}}{5/2} + \frac{2x^{-2}}{-2} + C_1 + C_2 + C_3 = \frac{x^6}{2} - \frac{8}{5}x^{5/2} - \frac{1}{x^2} + C$

5. $\int e^x dx$

$= e^x + C$

6.

$\int x^{-1} dx = \ln|x| + C$

Ex: **(a)** $\int \left(3x^4 - 5e^x + \frac{1}{x^2} - \frac{1}{x}\right) dx$

$= \int 3x^4 dx - \int 5e^x dx + \int x^{-2} dx - \int x^{-1} dx$

$= \frac{3}{5}x^5 - 5e^x + \frac{x^{-1}}{-1} - \ln|x| + C$

(b) $\int \left(6x^3 - \frac{3}{\sqrt{x}} + \sqrt{x}\right) dx$

$= 6\int x^3 dx - \int 3x^{-1/2} dx + \int x^{1/2} dx$

$= \frac{6x^4}{4} - \frac{3x^{1/2}}{1/2} + \frac{x^{3/2}}{3/2} + C$

$= \frac{3}{2}x^4 - 6x^{1/2} + \frac{2}{3}x^{3/2} + C$

(c) $\int \frac{x^4 - x + 1}{x^2} dx = \int \left(\frac{x^4}{x^2} - \frac{x}{x^2} + \frac{1}{x^2}\right) dx$

$= \int \left(x^2 - \frac{1}{x} + x^{-2}\right) dx$

$= \frac{x^3}{3} - \ln|x| - x^{-1} + C$

(d) $\int (x^2 + 2x)(x - 1) dx = \int x^3 + 2x^2 - x^2 - 2x \, dx$

$= \int x^3 + x^2 - 2x \, dx$

$= \frac{x^4}{4} + \frac{x^3}{3} - \frac{2x^2}{2} + C$