

Last Name: SOLUTIONS

First Name: _____

Student ID: _____

Test 1 (B)

The length of the test is 1hr and 45min. You may use a nonprogrammable, nongraphing scientific calculator. Remember to clearly show all of your work and to **use correct notation**. Do not use decimals unless otherwise stated.

Question 1. Find/evaluate the following:

$$(a) \text{ (3 marks)} \int \frac{3x^4 - x^2 + 6x}{x^3} dx = \int (3x - \frac{1}{x} + 6x^2) dx$$

$$= \frac{3}{2}x^2 - \ln|x| - \frac{6}{x} + C$$

$$(b) \text{ (4 marks)} \int \frac{3x^2 + 2x - 2}{(3x^3 + 3x^2 - 6x)^2} dx$$

$$= \int \frac{3x^2 + 2x - 2}{u^2} \cdot \frac{du}{3(3x^2 + 2x - 2)}$$

$$= \frac{1}{3} \int u^{-2} du = \frac{1}{3} \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{3} \cdot \frac{1}{3x^3 + 3x^2 - 6x} + C$$

$\text{LET } u = 3x^3 + 3x^2 - 6x$ $du = (9x^2 + 6x - 6)dx$ $\frac{du}{3(3x^2 + 2x - 2)} = dx$
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$$(c) \text{ (4 marks)} \int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int \frac{e^u}{\sqrt{x}} \cdot 2\sqrt{x} du$$

$$= 2 \int e^u du = 2e^u + C$$

$$= 2e^{\sqrt{x}} + C$$

LET $u = \sqrt{x}$

$$du = \frac{1}{2}x^{-1/2}dx$$

$$2\sqrt{x} du = dx$$

$$(d) \text{ (4 marks)} \int \frac{(\sqrt{x}-x)^2}{x} dx = \int \frac{x - 2x^{3/2} + x^2}{x} dx$$

$$= \int (1 - 2x^{1/2} + x) dx = x - \frac{4}{3}x^{3/2} + \frac{1}{2}x^2 + C$$

Question 2.

(a) (1 marks) Write the definition of the definite integral

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

(b) (3 marks) Using words describe

i) What Δx is.

Δx IS THE WIDTH OF AN APPROXIMATING RECTANGLE IN
THE RIEMANN SUM

ii) What $f(x_i) \Delta x$ is.

$f(x_i) \Delta x$ IS THE AREA OF AN APPROXIMATING RECTANGLE
ON THE SUBINTERVAL $[x_{i-1}, x_i]$ IN THE RIEMANN SUM.

iii) What the purpose of the limit in the definition is.

THE SUM OF APPROXIMATING RECTANGLES $\sum_{i=1}^n f(x_i) \Delta x$
IS AN APPROXIMATION OF THE "NET AREA" UNDER THE
FUNCTION $f(x)$ ON $[a, b]$. THIS APPROXIMATION GETS BETTER
AS n GETS LARGER (MORE RECTANGLES). THE LIMIT IN
THE DEFINITION GETS US THE EXACT AREA.

(c) (6 marks) Use the definition of the definite integral to find:

$$\int_1^5 (1+2x-x^2) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$$

- $\Delta x = \frac{b-a}{n} = \frac{5-1}{n} = \frac{4}{n}$
- $x_i = a + i\Delta x = 1 + \frac{4i}{n}$

- $f(x_i) = 1 + 2x_i - x_i^2 = 1 + 2\left(1 + \frac{4i}{n}\right) - \left(1 + \frac{4i}{n}\right)^2$
- $= 1 + 2 + \frac{8i}{n} - \left(1 + \frac{8i}{n} + \frac{16i^2}{n^2}\right) = 2 - \frac{16i^2}{n^2}$

- $f(x_i) \Delta x = \left(2 - \frac{16i^2}{n^2}\right)\left(\frac{4}{n}\right) = \frac{8}{n} - \frac{64i^2}{n^3}$

- $\sum_{i=1}^n \left(\frac{8}{n} - \frac{64i^2}{n^3}\right) = \frac{8}{n} \sum_{i=1}^n 1 - \frac{64}{n^3} \sum_{i=1}^n i^2 = \frac{8}{n} \cdot n - \frac{64}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$
- $= 8 - \frac{32}{3} \cdot \frac{2n^2 + 3n + 1}{n^2}$

- $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x = \lim_{n \rightarrow \infty} \left[8 - \frac{32}{3} \cdot \frac{2 + 3/n + 1/n^2}{1} \right]$

$$= 8 - \frac{32}{3} \cdot \frac{2 + 0 + 0}{1}$$

$$= 8 - \frac{64}{3} = -\frac{40}{3}$$

$$\therefore \int_1^5 (1+2x-x^2) dx = -\frac{40}{3}$$

Question 3. (5 marks) Let $f'(x) = \frac{1}{x} + \sqrt{x}$. Find $f(x)$ given $f(4) = 12$.

$$f(x) = \int f'(x) dx = \int \left(\frac{1}{x} + \sqrt{x}\right) dx = \ln|x| + \frac{2}{3}x^{3/2} + C$$

$$\begin{aligned}\therefore 12 &= f(4) = \ln 4 + \frac{2}{3}(4)^{3/2} + C \\ &= \ln 4 + \frac{16}{3} + C\end{aligned}$$

$$\Rightarrow 12 - \ln 4 - \frac{16}{3} = C$$

$$\frac{20}{3} - \ln 4 = C$$

$$\therefore f(x) = \ln|x| + \frac{2}{3}x^{3/2} + \frac{20}{3} - \ln 4$$

Question 4. Evaluate the following definite integrals:

$$(a) (5 \text{ marks}) \int_3^6 (x+2)\sqrt{x-2} dx$$

$$= \int_1^4 (u+2)\sqrt{u} du$$

$$= \int_1^4 (u+4)\sqrt{u} du = \int_1^4 (u^{3/2} + 4u^{1/2}) du$$

$$= \left[\frac{u^{5/2}}{5/2} + \frac{4u^{3/2}}{3/2} \right]_1^4 =$$

$$= \left[\frac{2}{5}(4)^{5/2} + \frac{8}{3}(4)^{3/2} \right] - \left[\frac{2}{5}(1)^{5/2} + \frac{8}{3}(1)^{3/2} \right]$$

$$= \cancel{\frac{2}{5}(64)} + \frac{8}{3}(8) - \frac{2}{5} - \cancel{\frac{8}{3}} = \frac{64}{5} - \frac{2}{5} + \frac{64}{3} - \frac{8}{3}$$

$$= \frac{62}{5} + \frac{56}{3} = \cancel{\frac{186}{15}} \quad \frac{466}{15}$$

$$(b) (5 \text{ marks}) \int_{\pi/12}^{\pi/6} \csc^2 3x dx$$

$$= \frac{1}{3} \int_{\pi/4}^{\pi/2} \csc^2 u du = \frac{1}{3} \left[-\cot u \right]_{\pi/4}^{\pi/2}$$

$$= -\frac{1}{3} \cot \frac{\pi}{2} - \left(-\frac{1}{3} \cot \left(\frac{\pi}{4} \right) \right)$$

$$= -\frac{1}{3}(0) + \frac{1}{3}(1)$$

$$= \frac{1}{3}$$

$$\text{LET } u = x-2$$

$$du = dx$$

$$u+2 = x$$

$$\text{IF } x = 3 \Rightarrow u = 1$$

$$x = 6 \Rightarrow u = 4$$

$$\text{LET } u = 3x$$

$$du = 3dx$$

$$\frac{du}{3} = dx$$

$$\text{IF } x = \frac{\pi}{12} \Rightarrow u = \frac{\pi}{4}$$

$$x = \pi/6 \Rightarrow u = \pi/2$$

Question 5. (5 marks) Given

$$\int_2^6 f(x) dx = 13, \quad \int_6^7 f(x) dx = -4 \quad \text{and} \quad \int_6^9 f(x) dx = 5$$

find the following (show your work for full marks):

$$\begin{aligned} \text{(a)} \quad \int_2^9 f(x) dx &= \int_2^6 f(x) dx + \int_6^9 f(x) dx \\ &= 13 + 5 \\ &= 18 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int_7^9 f(x) dx &= \int_6^9 f(x) dx - \int_6^7 f(x) dx \\ &= 5 - (-4) \\ &= 9 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int_6^2 f(x) dx &= - \int_2^6 f(x) dx \\ &= -13 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad \int_2^5 f(x) dx + \int_5^7 f(x) dx &= \int_2^7 f(x) dx \\ &= \int_2^6 f(x) dx + \int_6^7 f(x) dx \\ &= 13 + (-4) = 9 \end{aligned}$$

Question 6. (4 marks) Find the area of the region bounded between

$f(x) = x^3 - 3x^2 - 3x + 3$ and $g(x) = x + 3$. Sketch a graph to help solve the problem.

INTERSECTION POINTS

$$f(x) = g(x)$$

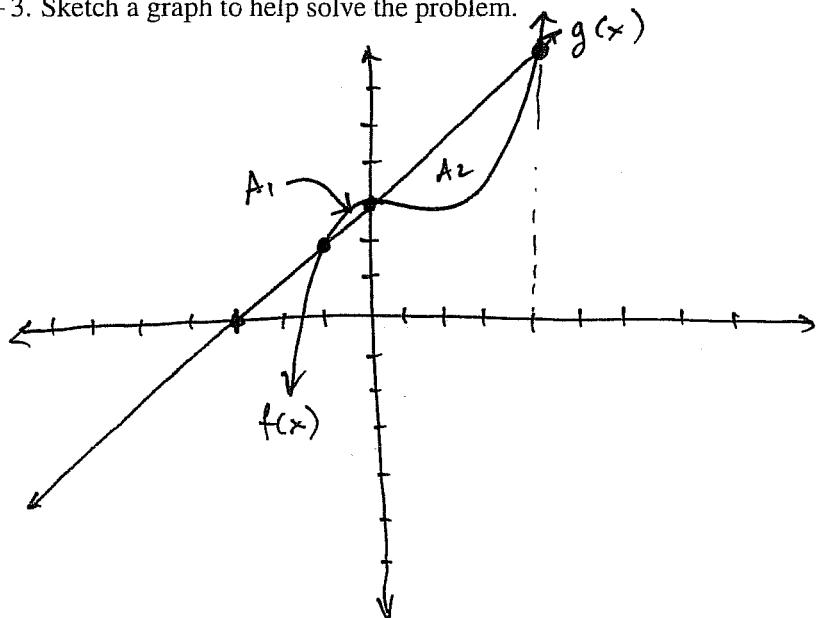
$$x^3 - 3x^2 - 3x + 3 = x + 3$$

$$x^3 - 3x^2 - 4x = 0$$

$$x(x^2 - 3x - 4) = 0$$

$$x(x-4)(x+1) = 0$$

$$\therefore x = -1, 0, 4$$



$$\therefore A = A_1 + A_2 = \int_{-1}^0 [f(x) - g(x)] dx + \int_0^4 [g(x) - f(x)] dx$$

$$\Rightarrow \int_{-1}^0 (x^3 - 3x^2 - 3x + 3) - (x+3) dx + \int_0^4 (x+3) - (x^3 - 3x^2 - 3x + 3) dx$$

$$= \int_{-1}^0 (x^3 - 3x^2 - 4x) dx + \int_0^4 (4x + 3x^2 - x^3) dx$$

$$= \left[\frac{x^4}{4} - x^3 - 2x^2 \right]_{-1}^0 + \left[2x^2 + x^3 - \frac{x^4}{4} \right]_0^4 = \left[0 - \left(\frac{(-1)^4}{4} - (-1)^3 - 2(-1)^2 \right) \right] + \left[2(4)^2 + (4)^3 - \frac{(4)^4}{4} - 0 \right]$$

$$= \left(-\frac{1}{4} + 1 + 2 \right) + (32 + 64 - 64)$$

$$= \frac{3}{4} + 32$$

$$= \frac{131}{4}$$