Differential Equations

A **Differential Equation** is an equation that involves an unknown function and its derivative(s).

For Example:

$$\frac{dy}{dt} = t\ln t, \qquad \frac{dy}{dx} + 3y = x^3, \qquad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + xy - 12 = 0$$

are examples of differential equations.

Differential equations appear in practically every branch of applied mathematics, and the study of these equations remains one of the most active areas in research in mathematics.

Solutions of Differential Equations A **solution** to a differential equation is any function that satisfies the differential equation.

Example: Show that the function $f(x) = ce^{-x} + x - 1$, where c is a constant, is a solution of the differential equation

$$y' + y = x$$

.

It can be shown that every solution of the differential equation $y' + y = x$ must have the form $y = ce^{-x} + x - 1$, where c is a constant. Therefore, this is a **general solution** of the differential equation $y' + y = x$. A solution obtained by assigning a specific value to the constant c is caled a **particular solution**.

Exapmle: Find the particular solution of the equation $y' + y = x$ that satisfies the condition $y(0) = 0$.

Example: Verify that $y = Ce^{2x} - 2x - 1$ is a general solution of the differential equation $y' - 2y - 4x = 0$. Find a particular solution of the differential equation that satisfies $y(0) = 3$

The Method of Separation of Variables

The **order** of a differential equation is the order of the highest derivative of the unknown function in the equation. For example

$$y' = xe^x \qquad \text{and} \quad \left(\frac{dy}{dx}\right)^2 + 3y = 5$$

are first-order differential equations whereas

$$3\frac{d^3y}{dx}^3 - \frac{dy}{dx} = 4y - 6x + 2$$

is a third-order differential equation.

If a first-order differential equatioin can be written as

$$\frac{dy}{dx} = f(x)g(y)$$

then we call this differential equation **separable**.

For example, the equation

$$\frac{dQ}{dt} = kCQ - kQ^2$$

is a separable equation because we can write

$$\frac{dQ}{dt} = kCQ - kQ^2$$

The differential equation

$$\frac{dy}{dx} = x^2y + y^2$$

is not separable.

We can solve separable first order equations by using the following method.

Method of Separation of Variables Suppose we are givena first-order separable differential equation in the form

$$\frac{dy}{dx} = f(x)g(y)$$

1) Separate the variables by writing the equation in the following form

$$\frac{1}{g(y)}\,dy = f(x)\,dx$$

2) Integrate each side of this equation with respect to the appropriate variable.

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

3) Solve for y.

Example: Find the general solution of the first order differential equation

$$y' = \frac{xy}{x^2+1}$$

Example: Find the solution of the differential equation

$$y' = 2x^2y + 2x^2$$

that satisfies the initial condition y(0) = 0

Example: Find the equation describing f given that the slope of the tangent line to the graph of f at any point (x, y) is given by the expression $-x/(2y)$ and the graph of f passes through the point $(1, 2)$.

Why does the method of separation of variables work?

Example: Solve the following first order differential eqation

$$y' = \frac{xy^2}{\sqrt{1+x^2}}$$

Example: Find the solution fo the initial value problem

$$y' = xe^{x^2}y; \quad y(0) = 1$$