Infinite Sequences

In mathematics, when we think of an **Infinite Sequence** (or **sequence** for short) we think of a list of numbers. Some examples of infinte sequences are:

1) $1, 2, 3, 4, 5, \ldots$

2) $4, 3, 2, 1, 0, -1, -2, \ldots$

3) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

4) $1, \frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$

We often denote the terms of a sequence by a_1, a_2, a_3, a_4,.... For short we may write

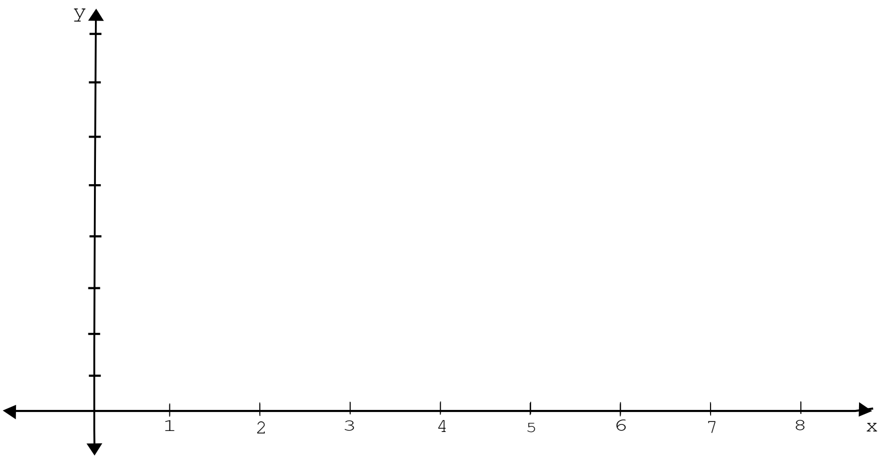
$\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$

to stand for the entire sequence.

Let's look more closely at the sequence

$$\{a_n\}_{n=1}^{\infty} = 1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \frac{1}{32}, \quad \ldots$$

We can graph this sequence



Written like this we can think of the sequence as a function where

$$f(1)=1, \quad f(2)=\frac{1}{2}, \quad f(3)=\frac{1}{4}, \quad f(4)=\frac{1}{8}, \quad f(5)=\frac{1}{16}, \quad f(6)=\frac{1}{32}, \quad \ldots$$

So even though we think of a sequence as a list of numbers, the actual definition of sequence is the following.

Definition: An **Infinite Sequence** $\{a_n\}$ is a function whose domain is the set of positive integers. The functional values a_1, a_2, a_3, a_4,... are the **terms** of the sequence, and the term a_n is called the ***n*th term** of the sequence.

Note: Sometimes sequences start at numbers other than 1. For example

$$\{a_n\}_{n=3}^{\infty} = a_3, a_4, a_5, \ldots$$

Notice that some sequences have a pattern. For Example:

1) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\ldots$

2) $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{6}\ldots$

For short we can write these sequences

Example: Find the *n*th term of each sequence.

1) $2, \ \frac{3}{\sqrt{2}}, \ \frac{4}{\sqrt{3}}, \ \frac{5}{\sqrt{4}} \cdots$

2) $-1, \ \frac{1}{8}, \ -\frac{1}{27}, \ \frac{1}{64} \cdots$

3) $1, \ -\frac{1}{2}, \ \frac{1}{3}, \ -\frac{1}{4} \cdots$

Limit of a Sequence

Let $\{a_n\}$ be a sequence. We say that the sequence $\{a_n\}$ **converges** and has the limit L, written

$$\lim_{n\to\infty} a_n = L$$

if the terms of the sequence, a_n, can be made as close to L as we please by taking n sufficiently large. If a sequence is not convergent, it is said to be **divergent**.

Example: Find the following limits (if they exist).

1) $\lim_{n\to\infty}\left(-\frac{2}{3}\right)^n$

2) $\lim_{n\to\infty}\sqrt[7]{n}$

3) $\lim_{n\to\infty} e^{-n}$

4) $\lim_{n\to\infty}(-2)^n$

Limit Properties of Sequences

Suppose

$$\lim_{n\to\infty} a_n \quad \text{and} \quad \lim_{n\to\infty} b_n$$

converge. Then

a) $\lim_{n\to\infty} ca_n$

b) $\lim_{n\to\infty}(a_n \pm b_n)$

c) $\lim_{n\to\infty} a_n b_n$

d) $\lim_{n\to\infty} \frac{a_n}{b_n}$

Examples: Find limits of the following sequences if they exist:

1) $a_n = \frac{5n}{n+3} - \frac{2}{n-1}$

2) $a_n = \frac{3 - 5^n}{5^n}$

3) $a_n = \frac{n}{n!}$

4) $a_n = \frac{6}{(-5)^n}$

5) $a_n = \frac{3n^2+1}{2n^2} - \frac{5}{n^3+1}$

6) $a_n = \frac{\sqrt[3]{n}+2}{4\sqrt[3]{n}-1}$

7) $a_n = 5 + \frac{2}{3^n}$

8) $a_n = \frac{3^n - 6}{3^n}$

9) $a_n = \frac{4n}{n!}$

10) $a_n = \frac{4^n}{n!}$