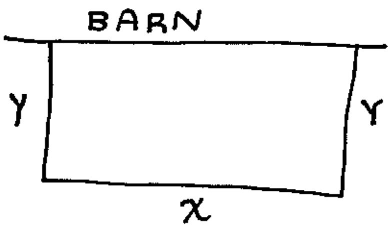
OPTIMIZATION PROBLEMS
In-class Assignment #8

1. a. A rectangular pen is built with one side against a barn. Two hundred metres of fencing are used for the other three sides of the pen. What dimensions maximize the area of the pen.

SOLUTION



$$x + 2y = 200 \text{ meters}$$
$$x = 200 - 2y$$

We want to maximize Area A:

$$A = xy$$
$$= (200-2y)y$$
$$= 200y - 2y^2$$

Find critical points

$$A' = 200 - 4y$$
$$= 4(50 - y)$$

$y = 50$ is a critical value

is it a max? check:

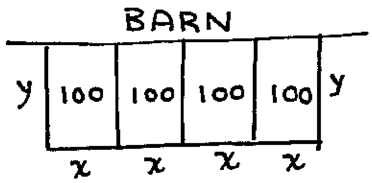
Intervals	$(0, 50)$	$(50, 100)$
Test pt.	1	51
Sign of A'	+	—
A incr. or decr.	↗	↘

When $y = 50$ Area A is a maximum.

$y = 50$ $\quad x = 200 - 2y$
$= 200 - 2(50)$
$= 100$

$$\boxed{\text{Dimensions to maximize Area are } x = 100,\ y = 50}$$

b. A rancher plans to make four identical & adjacent rectangular pens against a barn, each with an area of $100 m^2$. What are the dimensions of each pen that minimize the amount of fence that must be used?



SOLUTION

Amount of fence $:= P = 4x + 5y$

P is what we want to minimize

$$\text{Area} = 100 = xy$$

$$x = \frac{100}{y}$$

$$P = 4x + 5y$$
$$= 4\left(\frac{100}{y}\right) + 5y$$
$$= \frac{400}{y} + 5y$$

Find critical points:

$$P' = \frac{-400}{y^2} + 5$$

$$0 = \frac{-400}{y^2} + 5$$

$$\frac{400}{y^2} = 5$$

$$400 = 5y^2$$
$$y^2 = 80$$
$$y = \pm 8.95$$

only $y = 8.95$ m makes sense in this context

Is it a minimum?

Intervals	$(0, 8.95)$	$(8.95, 100)$
Test pt.	8	10
sign P'	$-$	$+$
P incr. or decr.	$\searrow$	$\nearrow$

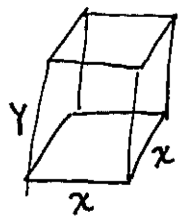
P is a minimum when $y = 8.95$m

$$x = \frac{100}{8.95} = 11.17\text{m}$$

$$\boxed{\begin{array}{l}\text{Dimensions to minimize amount of fence are} \\ x = 11.17\text{m} \ \& \ y = 8.95\text{m}\end{array}}$$

2. OF all boxes with a square base & a volume of $100 m^3$, which one has the minimum surface area?

SOLUTION



$$\text{Volume} = 100 m^3$$
$$x^2 y = 100$$
$$y = \frac{100}{x^2}$$

We want to minimize S.A.

$$SA = 2x^2 + 4xy$$
$$= 2x^2 + 4x\left(\frac{100}{x^2}\right)$$
$$= 2x^2 + \frac{400}{x}$$

Find critical points

$$SA' = 4x - \frac{400}{x^2}$$

$$0 = 4x - \frac{400}{x^2}$$

$$\frac{400}{x^2} = 4x$$
$$100 = x^3$$
$$x = \sqrt[3]{100}$$
$$\approx 4.64$$

Check if it is a minimum

INTERVAL	(0, 4.64)	(4.64, 10)
TEST PT.	1	5
Sign of SA'	—	+
incr or decr	↘	↗

It is a minimum when $x = 4.64 m$

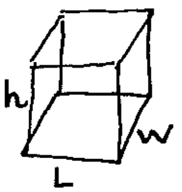
$$y = \frac{100}{(4.64)^2} = 4.64 m$$

$$\boxed{\text{The dimensions that will minimize the surface area } 4.64m \times 4.64m \times 4.64m}$$

(3) An airline policy states that all baggage must be box shaped with a sum of length, width & height not exceeding 108 in.
What are the dimensions & volume of a square-based box with the greatest volume under these conditions?

SOLUTION

DIMENSIONS



$L = w$

$$108 = L + w + h$$
$$108 = 2L + h$$
$$h = 108 - 2L$$

We want to maximize volume:

$$V = L^2 h$$
$$= L^2(108 - 2L)$$
$$V = 108L^2 - 2L^3$$

Find critical points

$$V' = 216L - 6L^2$$
$$= 6L(36 - L)$$

critical points $L = 0$, $L = 36$

only $36 = L$ makes sense

Check if it is a maximum

Intervals	$(0, 36)$	$(36, \ )$
Test pt	1	
sign of V'	$+$	$-$
V incr/decr.	↗	↘

Volume V is a maximum when $L = 36$

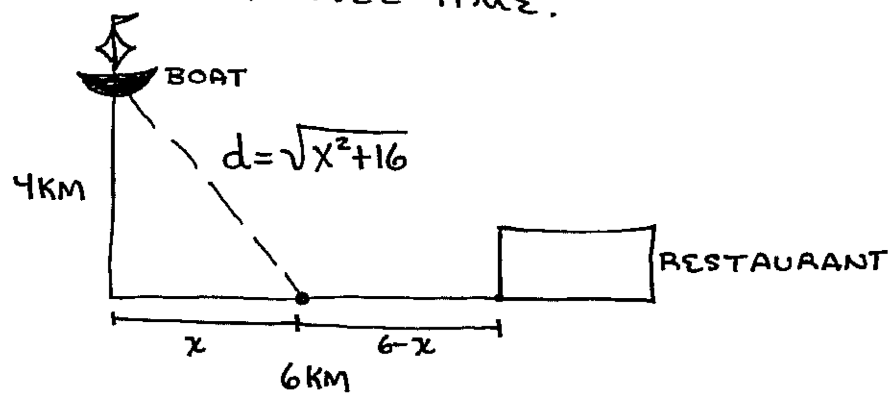
$w = 36$

& $h = 108 - 2(36)$
$= 36$

$$\boxed{\text{Dimensions } 36\text{in} \times 36\text{in} \times 36\text{in} \quad V = 36^3 \text{ in}^3 = 46{,}656 \text{ in}^3}$$

(4) A boat on the ocean is 4km from the nearest point on a straight shoreline; that point is 6km from a restaurant on the shore. A woman plans to row the boat straight to a point on the shore & then walk along the shore to the restaurant.

If she walks 3km/hr & rows 2km/hr at which point on the shore should she land to minimize the total travel time.



SOLUTION

We want to minimize travel time

$$T = \text{Time on boat} + \text{Time on shore}$$

$$= \frac{\text{Distance (boat)}}{\text{Speed (boat)}} + \frac{\text{Distance (shore)}}{\text{Speed (shore)}}$$

$$= \frac{\sqrt{x^2+16}}{2} + \frac{6-x}{3}$$

$$= \tfrac{1}{2}(x^2+16)^{1/2} + \tfrac{1}{3}(6-x)$$

Find critical pts:

$$T' = \tfrac{1}{4}(x^2+16)^{-1/2}(2x) - \tfrac{1}{3}$$

$$= \frac{x}{2\sqrt{x^2+16}} - \frac{1}{3}$$

$$0 = \frac{x}{2\sqrt{x^2+16}} - \frac{1}{3}$$

$$\frac{1}{3} = \frac{x}{2\sqrt{x^2+16}}$$

$$2\sqrt{x^2+16} = 3x$$

$$\left(2\sqrt{x^2+16}\right)^2 = (3x)^2$$

$$4(x^2+16) = 9x^2$$

$$64 = 5x^2$$

$$x^2 = 12.8$$

$$x = \pm 3.578 \text{ KM}$$

IS $x = 3.578$ A MINIMUM

INTERVALS	$(0, 3.578)$	$(3.578, 6)$
TEST PT	1	4
Sign OF T'	$-$	$+$
T incr./decr.	↘	↗

$$\boxed{x = 3.578 \text{ Km is A MINIMUM}}$$

THE MINIMAL TRAVEL TIME WILL be:

$$T = \frac{\sqrt{x^2+16}}{2} + \frac{6-x}{3}$$

$$= \frac{\sqrt{(3.578)^2+16}}{2} + \frac{6-3.578}{3}$$

$$= 2.68 \text{ hr} + 0.81 \text{ hr}$$

$$= 3.49 \text{ hrs}$$